# Topic 2: Functions

## Concepts

### Essential understandings

Models are depictions of real-life events using expressions, equations or graphs while a function is defined as a relation or expression involving one or more variables. Creating different representations of functions to model the relationships between variables, visually and symbolically as graphs, equations and tables represents different ways to communicate mathematical ideas.

**Suggested concepts embedded in this topic:**

Representation, relationships, space, quantity, equivalence.

**AHL:** Systems, patterns.

**Content-specific conceptual understandings:**

- Different representations of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships.
- The parameters in a function or equation correspond to geometrical features of a graph and can represent physical quantities in spatial dimensions.
- Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.
- Our spatial frame of reference affects the visible part of a function and by changing this “window” can show more or less of the function to best suit our needs.
- Equivalent representations of quadratic functions can reveal different characteristics of the same relationship.
- Functions represent mappings that assign to each value of the independent variable (input) one and only one dependent variable (output).

AHL

- Extending results from a specific case to a general form can allow us to apply them to a larger system.

- Patterns can be identified in behaviours which can give us insight into appropriate strategies to model or solve them.
- The intersection of a system of equations may be represented graphically and algebraically and represents the solution that satisfies the equations.

## SL content

Recommended teaching hours: 21

The aim of the SL content in the functions topic is to introduce students to the important unifying theme of a function in mathematics and to apply functional methods to a variety of mathematical situations.

Throughout this topic students should be given the opportunity to use technology, such as graphing packages and graphing calculators to develop and apply their knowledge of functions, rather than using elaborate analytic techniques.

On examination papers:

- questions may be set requiring the graphing of functions that do not explicitly appear on the syllabus
- the domain will be the largest possible domain for which a function is defined unless otherwise stated; this will usually be the real numbers

Sections SL2.1 to SL2.4 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

### SL 2.1

| Content | Guidance, clarification and syllabus links |
|---|---|
| Different forms of the equation of a straight line.<br><br>Gradient; intercepts.<br><br>Lines with gradients $m_1$ and $m_2$<br><br>Parallel lines $m_1 = m_2$.<br><br>Perpendicular lines $m_1 \times m_2 = -1$. | $y = mx + c$ (gradient-intercept form).<br><br>$ax + by + d = 0$ (general form).<br><br>$y - y_1 = m(x - x_1)$ (point-gradient form).<br><br>Calculate gradients of inclines such as mountain roads, bridges, etc. |

Connections

**Other contexts:** Gradients of mountain roads, gradients of access ramps.

**Links to other subjects:** Exchange rates and price and income elasticity, demand and supply curves (economics); graphical analysis in experimental work (sciences group subjects).

**TOK**: Descartes showed that geometric problems could be solved algebraically and vice versa. What does this tell us about mathematical representation and mathematical knowledge?

| Download connections template |
|---|

## SL 2.2

| Content | Guidance, clarification and syllabus links |
|---|---|
| Concept of a function, domain, range and graph.<br><br>Function notation, for example<br>$f(x), \quad v(t), \quad C(n)$.<br><br>The concept of a function as a mathematical model. | **Example:** $f(x)=\sqrt{2-x}$, the domain is $x \le 2$, range is $f(x) \ge 0$.<br><br>A graph is helpful in visualizing the range. |
| Informal concept that an inverse function reverses or undoes the effect of a function.<br><br>Inverse function as a reflection in the line $y = x$, and the notation $f^{-1}(x)$. | **Example:** Solving $f(x)=10$ is equivalent to finding $f^{-1}(10)$.<br><br>Students should be aware that inverse functions exist for one to one functions; the domain of $f^{-1}(x)$is equal to the range of $f(x)$. |

Connections

**Other contexts:** Temperature and currency conversions**.**

**Links to other subjects:** Currency conversions and cost functions (economics and business management); projectile motion (physics).

**Aim 8:** What is the relationship between real-world problems and mathematical models?

**International-mindedness**: The development of functions by Rene Descartes (France), Gottfried Wilhelm Leibnitz (Germany) and Leonhard Euler (Switzerland); the notation for functions was developed by a number of different mathematicians in the 17th and 18th centuries–how did the notation we use today become internationally accepted?

**TOK**: Do you think mathematics or logic should be classified as a language?

Download connections template

**SL 2.3**

| Content | Guidance, clarification and syllabus links |
|---|---|
| The graph of a function; its equation $y = f(x)$. | Students should be aware of the difference between the command terms “draw” and “sketch”. |
| Creating a sketch from information given or a context, including transferring a graph from screen to paper.<br><br>Using technology to graph functions including their sums and differences. | All axes and key features should be labelled.<br><br>This may include functions not specifically mentioned in topic 2. |

Connections

**Links to other subjects:** Sketching and interpreting graphs (sciences group subjects, geography, economics).

**TOK**: Does studying the graph of a function contain the same level of mathematical rigour as studying the function algebraically? What are the advantages and disadvantages of having different forms and symbolic language in mathematics?

Download connections template

## SL 2.4

| Content | Guidance, clarification and syllabus links |
| --- | --- |
| Determine key features of graphs. | Maximum and minimum values; intercepts; symmetry; vertex; zeros of functions or roots of equations; vertical and horizontal asymptotes using graphing technology. |
| Finding the point of intersection of two curves or lines using technology. | |

Connections

**Links to other subjects:** Identification and interpretation of key features of graphs (sciences group subjects, geography, economics); production possibilities curve model, market equilibrium (economics).

**International-mindedness**: Bourbaki group analytical approach versus the Mandlebrot visual approach.

**Use of technology:** Graphing technology with sliders to determine the effects of altering parameters and variables.

| |
| --- |
| Download connections template |

### SL 2.5

| Content | Guidance, clarification and syllabus links |
| --- | --- |
| Composite functions. | $(f \circ g)(x) = f(g(x))$ |
| Identity function. Finding the inverse function $f^{-1}(x)$. | $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$<br><br>The existence of an inverse for one-to-one functions.<br><br>**Link to**: concept of inverse function as a reflection in the line $y = x$ (SL 2.2). |

Connections

**TOK**: Do you think mathematics or logic should be classified as a language?

Download connections template

### SL 2.6

| Content | Guidance, clarification and syllabus links |
| --- | --- |
| The quadratic function $f(x) = ax^2 + bx + c$: its graph, $y$-intercept $(0, c)$. Axis of symmetry.<br><br>The form $f(x) = a(x - p)(x - q)$, $x$-intercepts $(p, 0)$ and $(q, 0)$.<br><br>The form $f(x) = a(x - h)^2 + k$, vertex $(h, k)$. | A quadratic graph is also called a parabola.<br><br>**Link to**: transformations (SL 2.11).<br><br>Candidates are expected to be able to change from one form to another. |

Connections

**Links to other subjects:** Kinematics, projectile motion and simple harmonic motion (physics).

**TOK**: Are there fundamental differences between mathematics and other areas of knowledge? If so, are these differences more than just methodological differences?

Download connections template

**SL 2.7**

| Content | Guidance, clarification and syllabus links |
|---|---|
| Solution of quadratic equations and inequalities.<br><br>The quadratic formula. | Using factorization, completing the square (vertex form), and the quadratic formula.<br><br>Solutions may be referred to as roots or zeros. |
| The discriminant $\Delta = b^2 - 4ac$ and the nature of the roots, that is, two distinct real roots, two equal real roots, no real roots. | **Example**: For the equation $3kx^2 + 2x + k = 0$, find the possible values of $k$, which will give two distinct real roots, two equal real roots or no real roots. |

Connections

**Links to other subjects:** Projectile motion and energy changes in simple harmonic motion (physics); equilibrium equations (chemistry).

**International-mindedness**: The Babylonian method of multiplication: $ab = \frac{(a+b)^2 - a^2 - b^2}{2}$. Sulba Sutras in ancient India and the Bakhshali Manuscript contained an algebraic formula for solving quadratic equations.

**TOK**: What are the key concepts that provide the building blocks for mathematical knowledge?

**Use of technology:** Dynamic graphing software with a slider.

**Enrichment:** Deriving the quadratic formula by completing the square.

# Syllabus

Download connections template

### SL 2.8

| Content | Guidance, clarification and syllabus links |
|---|---|
| The reciprocal function $f(x)=\frac{1}{x},\ x\neq 0$: its graph and self-inverse nature. | |
| Rational functions of the form $f(x)=\frac{ax+b}{cx+d}$ and their graphs.<br><br>Equations of vertical and horizontal asymptotes. | Sketches should include all horizontal and vertical asymptotes and any intercepts with the axes.<br><br>**Link to:** transformations (SL2.11).<br><br>Vertical asymptote: $x=-\frac{d}{c}$;<br><br>Horizontal asymptote: $y=\frac{a}{c}$. |

Connections

**International-mindedness**: The development of functions, Rene Descartes (France), Gottfried Wilhelm Leibniz (Germany) and Leonhard Euler (Switzerland).

**TOK**: What are the implications of accepting that mathematical knowledge changes over time?

Download connections template

### SL 2.9

| Content | Guidance, clarification and syllabus links |
| --- | --- |
| Exponential functions and their graphs:<br><br>$f(x)=a^x,\ a>0,\ f(x)=e^x$<br><br>Logarithmic functions and their graphs:<br><br>$f(x)=\log_a x,\ x>0,\ f(x)=\ln x,\ x>0.$ | **Link to**: financial applications of geometric sequences and series (SL 1.4).<br><br>Relationships between these functions:<br><br>$a^x=e^{x\ln a};\ \log_a a^x=x,\ a, \quad x>0, \quad a\neq 1$<br><br>Exponential and logarithmic functions as inverses of each other. |

Connections

**Links to other subjects:** Radioactive decay, charging and discharging capacitors (physics); first order reactions and activation energy (chemistry); growth curves (biology).

**Aim 8:** The phrase “exponential growth” is used popularly to describe a number of phenomena. Is ths a misleading use of a mathematical term?

**TOK**: What role do “models” play in mathematics? Do they play a different role in mathematics compared to their role in other areas of knowledge?

| Download connections template |
| --- |

**SL 2.10**

| Content | Guidance, clarification and syllabus links |
|---|---|
| Solving equations, both graphically and analytically. | **Example:** $e^{2x}-5e^{x}+4=0$.<br><br>**Link to:** function graphing skills (SL 2.3). |
| Use of technology to solve a variety of equations, including those where there is no appropriate analytic approach. | **Examples:** $e^{x}=\sin x$<br>$x^{4}+5x-6=0$ |
| Applications of graphing skills and solving equations that relate to real-life situations. | **Link to:** exponential growth (SL 2.9) |

Connections

**Other contexts:** Radioactive decay and population growth and decay, compound interest, projectile motion, braking distances.

**Links to other subjects:** Radioactive decay (physics); modelling (sciences group subjects); production possibilities curve model (economics).

**TOK**: What assumptions do mathematicians make when they apply mathematics to real-life situations?

| |
|---|
| Download connections template |

### SL 2.11

| Content | Guidance, clarification and syllabus links |
|---|---|
| Transformations of graphs.<br><br>Translations: $y = f(x) + b$; $y = f(x - a)$.<br><br>Reflections (in both axes):<br>$y = -f(x)$; $y = f(-x)$.<br><br>Vertical stretch with scale factor $p$: $y = pf(x)$.<br><br>Horizontal stretch with scale factor $\frac{1}{q}$: $y = f(qx)$. | Students should be aware of the relevance of the order in which transformations are performed.<br><br>Dynamic graphing packages could be used to investigate these transformations. |
| Composite transformations. | **Example:** Using $y = x^2$ to sketch $y = 3x^2 + 2$<br><br>**Link to:** composite functions (SL2.5).<br><br>**Not required at SL**: transformations of the form $f(ax + b)$. |

Connections

**Links to other subjects:** Shift in supply and demand curves (Economics); induced emf and simple harmonic motion (physics).

Download connections template

## AHL content

Recommended teaching hours: 11

The aim of the AHL functions topic is to extend and build upon the aims, concepts and skills from the SL content. It introduces students to useful techniques for finding and using roots of polynomials,