The derivative of exponential functions b^x ($b > 0, b \neq 1$)

In the previous section, we established the derivative rule for the exponential function when the base b is equal to the quite special irrational number e: $\frac{d}{dx}(e^x) = e^x$. What about an exponential function with a base b other than e? Remember that b must be positive and not equal to 1. We can use the laws of logarithms to write b^x in terms of e^x. Recall from Section 4.5 that $b^{\log_b x} = x$, and if $b = e$ then $e^{\ln x} = x$. Hence, $b^x = e^{x \ln b}$ because $e^{x \ln b} = e^{\ln(b^x)} = b^x$. We can now find the derivative of b^x by applying the chain rule to its equivalent expression $e^{x \ln b}$.

$y = f(g(x)) = e^{x \ln b} \Rightarrow$ 'outside' function is $f(u) = e^u \quad f'(u) = e^u$

$\Rightarrow$ 'inside' function is $g(x) = x \ln b \quad g'(x) = \ln b$ [$\ln b$ is a constant]

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = e^{x \ln b} \cdot \ln b$$

$$\frac{dy}{dx} = b^x \ln b$$

Therefore, $\frac{d}{dx}(b^x) = b^x \ln b$.

This result agrees with the fact that $\frac{d}{dx}(e^x) = e^x$. Using this 'new' general rule, $\frac{d}{dx}(b^x) = b^x \ln b$, then $\frac{d}{dx}(e^x) = e^x \ln e$. Since $\ln e = 1$, then $\frac{d}{dx}(e^x) = e^x$.

The derivative of b^x

For $b > 0$ and $b \neq 1$, if $f(x) = b^x$, then $f'(x) = b^x \ln b$. Or, in Leibniz notation, $\frac{d}{dx}(b^x) = b^x \ln b$.

This result now answers the question we posed near the end of Section 13.1. In that section, we used the definition of the derivative to determine that the derivative of the general exponential function $f(x) = b^x$ is $b^x \cdot f'(0)$, where $f'(0)$ is the slope of the graph at $x = 0$. From our result above, we can see that for a specific base b, the slope of the curve $y = b^x$ when $x = 0$ is $\ln b$. The first screen image below is from Section 13.1 and shows the value of $f'(0)$ for $b = 2, 3$ and $\frac{1}{2}$. Evaluating $\ln 2$, $\ln 3$ and $\ln(\frac{1}{2})$ on a GDC confirms that $f'(0)$ is equal to $\ln b$.

```
nDeriv(2^X,X,0)
      .6931472361
nDeriv(3^X,X,0)
       1.09861251
nDeriv((1/2)^X,X
,0)
     -.6931472361
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ln(2)
      .6931471806
ln(3)
      1.098612289
ln(1/2)
     -.6931471806
```

Exercise 13.2

1 Find the derivative of each function.

a) $y = (3x - 8)^4$
b) $y = \sqrt{1 - x}$
c) $y = \ln(x^2)$
d) $y = 2\sin\left(\frac{x}{2}\right)$
e) $y = (x^2 + 4)^{-2}$
f) $y = e^{-3x}$
g) $y = \frac{1}{\sqrt{x + 2}}$
h) $y = \cos^2 x$
i) $y = e^{x^2} - 2x$
j) $y = \frac{1}{3x^2 - 5x + 7}$
k) $y = \sqrt[3]{2x + 5}$
l) $y = \ln(x^2 - 9)$

2 Find the equation of the line tangent to the given curve at the specified value of x. Express the equation exactly in the form $y = mx + c$.

a) $y = (2x^2 - 1)^3 \quad x = -1$
b) $y = \sqrt{3x^2 - 2} \quad x = 3$
c) $y = \sin 2x \quad x = \pi$

3 An object moves along a line so that its position s relative to a starting point at any time $t \geqslant 0$ is given by $s(t) = \cos(t^2 - 1)$.

a) Find the velocity of the object as a function of t.
b) What is the object's velocity at $t = 0$?
c) In the interval $0 < t < 2.5$, find any times (values of t) for which the object is stationary.
d) Describe the object's motion during the interval $0 < t < 2.5$.

4 In a) – f), find $\frac{dy}{dx}$. Use your GDC to check your answer.

a) $y = \sqrt{x^2 + 2x + 1}$
b) $y = \frac{1}{\sin x}$
c) $y = (x + \sqrt{x})^3$
d) $y = e^{\cos x}$
e) $y = (\ln x)^2$
f) $y = \frac{3}{\sqrt{2x + 1}}$

For questions 5–7, find the equation of a) the tangent, and b) the normal to the curve at the given point.

5 $y = \frac{2}{x^2 - 8}$ at (3, 2)

6 $y = \sqrt{1 + 4x}$ at (2, 3)

7 $y = \ln(4x - 3)$ at (1, 0)

8 Consider the exponential function $f(x) = 2^x$.

a) Find $f'(x)$.
b) Find the equation of the tangent to the graph of f at the point (0, 1).
c) Explain why the graph of f has no stationary points.

9 Consider the trigonometric curve $y = \sin\left(2x - \frac{\pi}{2}\right)$.

a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
b) Find the exact coordinates of any inflexion points for the curve in the interval $0 < x < \pi$.