Math SL PROBLEM SET 85

1. (CI) Let $f(x) = x^2 - 4x + 5$.

   a. Find the equation of the axis of symmetry of the graph of f .

   The function can also be expressed in the form $f(x) = (x - h)^2 + k$.

   b. (i) Write down the value of h .
      (ii) Find the value of k .

2. (CI) Let $\sin\theta = \frac{\sqrt{5}}{3}$, where θ is acute.

   a. Find $\cos\theta$.
   b. Find $\cos 2\theta$.

3. (CI) The values in the fourth row of Pascal's triangle are: 1 4 6 4 1

   a. Write down the values in the fifth row of Pascal's triangle.
   b. Hence or otherwise, find the term in x^3 in the expansion of $(2x + 3)^5$.

4. (CI) The position vectors of points P and Q are $\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $7\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}$ respectively.

   a. Find a vector equation of the line that passes through P and Q.
   b. The line through P and Q is perpendicular to the vector $2\boldsymbol{i} + n\boldsymbol{k}$. Find the value of n .

5. (CI) Events A and B are independent with $P(A \cap B) = 0.2$ and $P(A' \cap B) = 0.6$.

   a. Find $P(B)$.
   b. Find $P(A \cup B)$.

6. (CI) Let $f'(x) = \sin^3(2x)\cos(2x)$. Find $f(x)$, given that $f\left(\frac{\pi}{4}\right) = 1$.

7. (CI) Let $f(x) = m - \frac{1}{x}$ for $x \neq 0$. The line $y = x - m$ intersects the graph of f in two distinct points. Find the possible values of m .

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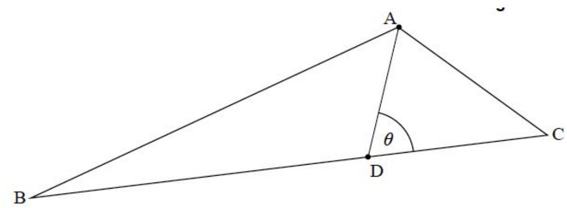
8. (CI) Let $\vec{OA} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$ and $\vec{OB} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$

   a. (i) Find AB
      (ii) Find AB

   The point C is such that $\vec{AC} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$.

   b. Show that the coordinates of C are $(-2, 1, 3)$.

   The following diagram shows triangle ABC. Let D be a point on [BC], with acute $\angle ADC = \theta$.

   c. Write down an expression in terms of θ for
      i. angle ADB;
      ii. area of triangle ABD.

   d. Given that $\frac{area\ \Delta ABD}{area\ \Delta ACD} = 3$, show that $\frac{BD}{BC} = \frac{3}{4}$.

   e. Hence or otherwise, find the coordinates of point D.

9. (CI) The first 2 terms of an infinite geometric sequence, in order, are $2\log_2 x$, $\log_2 x$, where $x>0$.

   a. Find r.
   b. Show that the sum of the infinite sequence is $4\log_2 x$.

   The first 3 terms of an arithmetic sequence, in order, are $\log_2 x$, $\log_2\left(\frac{x}{2}\right)$, $\log_2\left(\frac{x}{4}\right)$, where $x>0$.

   c. Find d, giving your answer as an integer.

   Let S_{12} be the sum of the first 12 terms of the arithmetic sequence.

   d. Show that $S_{12} = 12\log_2 x - 66$.
   e. Given that S_{12} is equal to half the sum of the infinite geometric sequence, find x, giving your answer in the form 2^p, where $p \in \mathbb{Q}$.

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10. (CI) Let $f(x) = \cos x$.

   a. (i) Find the first four derivatives of $f(x)$.
      (ii) Find $f^{(19)}(x)$.

   Let $g(x) = x^k$, where $k \in \mathbb{Z}^+$.

   b. (i) Find the first three derivatives of $g(x)$.
      (ii) Given that $g^{(19)}(x) = \frac{k!}{(k-p)!}(x^{k-19})$, find p .

   Let $k = 21$ and $h(x) = (f^{(19)}(x) \times g^{(19)}(x))$.

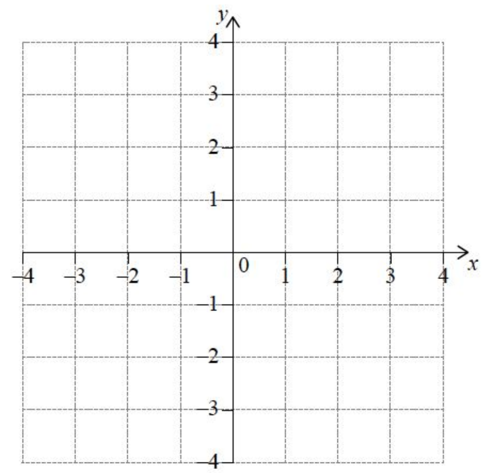
   c. (i) Find $h'(x)$.
      (ii) Hence, show that $h'(\pi) = \frac{-21!}{2}\pi^2$.

11. (CA) Let $f(x) = x^2 + 2x + 1$ and $g(x) = x - 5$, for $x \in \mathbb{R}$.

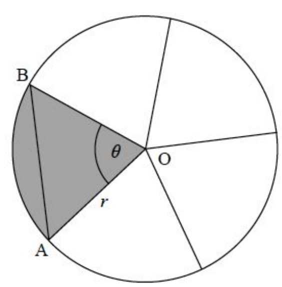
   a. Find $f(8)$.
   b. Find $(g \circ f)(x)$.
   c. Solve $(g \circ f)(x) = 0$.

12. (CA) Let $f(x) = 0.225x^3 - 2.7x$, for $-3 \le x \le 3$. There is a local minimum point at A.

   a. Find the coordinates of A.
   b. On the following grid,
      i. sketch the graph of f, clearly indicating the point A;
      ii. sketch the tangent to the graph of f at A.



13. (CA) The following diagram shows a circle, centre O and radius r mm . The circle is divided into five equal sectors. One sector is OAB, and $\angle AOB = \theta$.



   a. Write down the exact value of θ in radians.

   The area of sector AOB is 20π mm^2 .

   b. Find the value of r.
   c. Find AB

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14. (CA) Let $f(x) = xe^{-x}$ and $g(x) = -3f(x) + 1$. The graphs of f and g intersect at $x = p$ and $x = q$, where $p < q$.

   a. Find the value of p and of q .
   b. Hence, find the area of the region enclosed by the graphs of f and g .

15. (CA) A jar contains 5 red discs, 10 blue discs and m green discs. A disc is selected at random and replaced. This process is performed four times.

   a. Write down the probability that the first disc selected is red.
   b. Let X be the number of red discs selected. Find the smallest value of m for which $\text{Var}(X) < 0.6$.

16. (CA) Ten students were surveyed about the number of hours, x , they spent browsing the Internet during week 1 of the school year. The results of the survey are given below.

$$\sum_{i=1}^{10} x_i = 252, \sigma = 5 \text{ and median} = 27 .$$

   a. Find the mean number of hours spent browsing the Internet.
   b. During week 2, the students worked on a major project and they each spent an additional five hours browsing the Internet. For week 2, write down
      i. the mean;
      ii. the standard deviation.

   c. During week 3 each student spent 5% less time browsing the Internet than during week 1. For week 3, find
      i. the median;
      ii. the variance.

   d. During week 4, the survey was extended to all 200 students in the school. The results are shown in the cumulative frequency graph on the following page.
      i. Find the number of students who spent between 25 and 30 hours browsing the Internet.
      ii. Given that 10 % of the students spent more than k hours browsing the Internet, find the maximum value of k .

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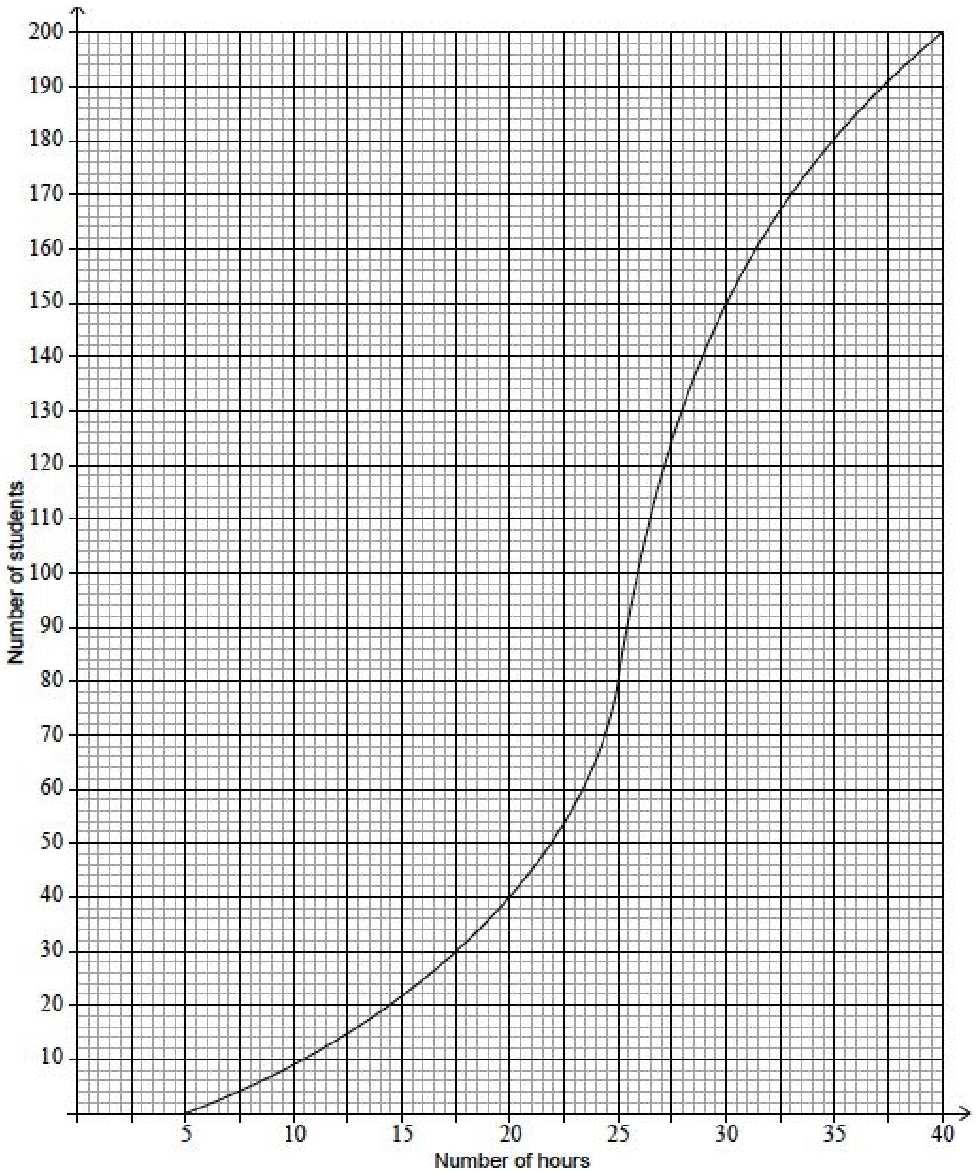
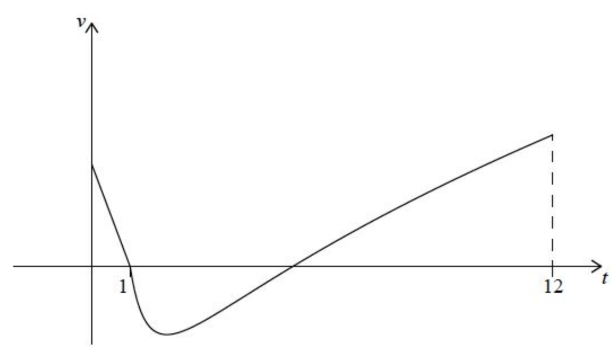


17. (CA) A particle P starts from a point A and moves along a horizontal straight line. Its velocity v cm s^{-1} after t seconds is given by

$$v(t) = \begin{cases} -2t + 2, \text{ for } 0 \le t \le 1 \\ 3\sqrt{t} + \frac{4}{t^2} - 7, \text{ for } 1 \le t \le 12 \end{cases}$$

The following diagram shows the graph of v.



a. Find the initial velocity of P.

P is at rest when $t = 1$ and $t = p$.

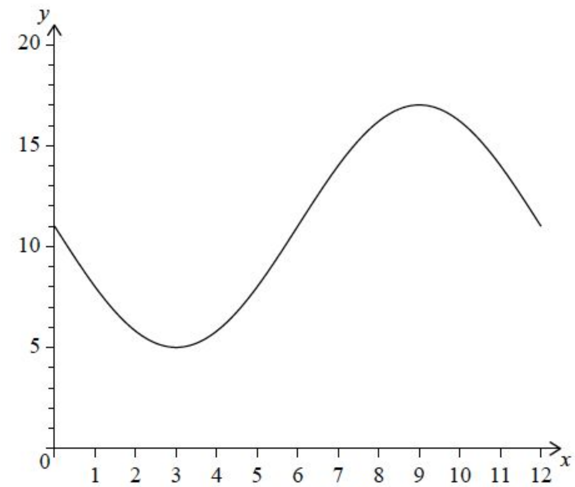
b. Find the value of p .

c. When $t = q$, the acceleration of P is zero.

i. Find the value of q.
ii. Hence, find the speed of P when $t = q$.

d. (i) Find the total distance travelled by P between $t = 1$ and $t = p$.
(ii) Hence or otherwise, find the displacement of P from A when $t = p$.

18. (CA) The following diagram shows the graph of $f(x) = a \sin bx + c$, for $0 \le x \le 12$. The graph of f has a minimum point at (3, 5) and a maximum point at (9, 17) .



a. (i) Find the value of c .
(ii) Show that
(iii) Find the value of a.

The graph of g is obtained from the graph of f by a translation of $\begin{pmatrix} k \\ 0 \end{pmatrix}$. The maximum point on the graph of g has coordinates (11.5, 17) .

b. (i) Write down the value of k .
(ii) Find $g(x)$.

The graph of g changes from concave-up to concave-down when $x = w$.

c. (i) Find w .
(ii) Hence or otherwise, find the maximum positive rate of change of g .

(November 2016)