3.2 Properties of Graphs of Quadratic Relations

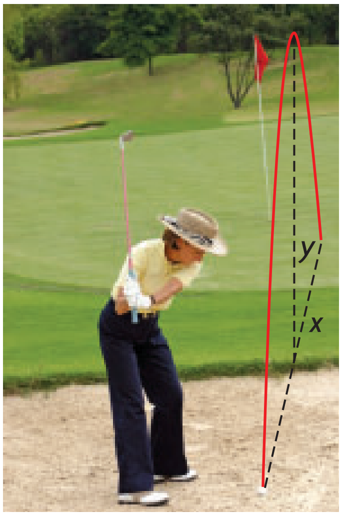
YOU WILL NEED

- grid paper
- ruler
- graphing calculator

GOAL

Describe the key features of the graphs of quadratic relations, and use the graphs to solve problems.

LEARN ABOUT the Math



Grace hits a golf ball out of a sand trap, from a position that is level with the green. The path of the ball is approximated by the equation $y = -x^2 + 5x$, where x represents the horizontal distance travelled by the ball in metres and y represents the height of the ball in metres.

Health *Connection*

Ultraviolet sun rays can damage the skin and cause skin cancer. Wearing a hat with a broad brim around the entire hat provides protection.

? What is the greatest height reached by the ball and how far away does it land?

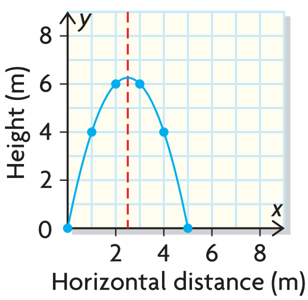
EXAMPLE 1 **Reasoning** from a table of values and a graph of a quadratic model

Determine the greatest height of the ball and the distance away that it lands.

Erika's Solution

x	0	1	2	3	4	5
y	0	4	6	6	4	0

I made a table of values. I used only positive values of x since the ball moves forward, not backward, when hit. When I reached a y-value of 0, I stopped. I assumed that the ball would not go below ground level.



I used my table of values to sketch the graph. I knew that the graph was a parabola, since the degree of the equation is 2. The parabola has a vertical line of symmetry that appears to pass through $x = 2.5$.

When $y = 0$, $x = \frac{0 + 5}{2} = 2.5$.

When $y = 4$, $x = \frac{1 + 4}{2} = 2.5$.

When $y = 6$, $x = \frac{2 + 3}{2} = 2.5$.

The equation of the axis of symmetry is $x = 2.5$.

I noticed that the points on the parabola with the same y-coordinate were the same distance from the line of symmetry. I reasoned that the **axis of symmetry** is the perpendicular bisector of any line segment joining points with the same y-coordinates. The means of the x-coordinates of these points give the equation of the axis of symmetry.

axis of symmetry

a line that separates a 2-D figure into two identical parts; if the figure is folded along this line, one of these parts fits exactly on the other part

$y = -x^2 + 5x$
$y = -(2.5)^2 + 5(2.5)$
$y = -6.25 + 12.5$
$y = 6.25$

The coordinates of the vertex are (2.5, 6.25).

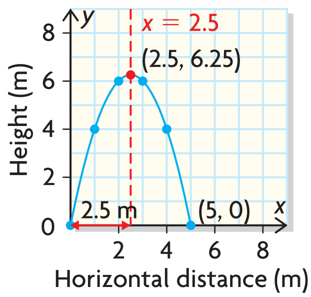
I saw that the **vertex** intersects the axis of symmetry, so its x-coordinate is 2.5. I substituted this value of x into the equation to get the **maximum value**.

vertex

the point of intersection of a parabola and its axis of symmetry

maximum value

the greatest value of the dependent variable in a relation



From my graph, I saw that $y = 0$ when $x = 5$. So, the ball lands 5 m away from where it was hit.

The ball's greatest height is 6.25 m. This occurs at a horizontal distance of 2.5 m from the starting point. The ball lands 5 m from the starting point.

Reflecting

A. Was the table of values or the graph more useful for determining the maximum height of the ball and the distance between where it was hit and where it landed? Explain.

B. How is the x-value at the maximum height of the ball related to the x-value of the point where the ball touches the ground?

C. Is it possible to predict whether a quadratic relation has a maximum value if you know the equation of the relation? Explain.

APPLY the Math

EXAMPLE 2 | **Selecting a table of values strategy** to graph a quadratic relation

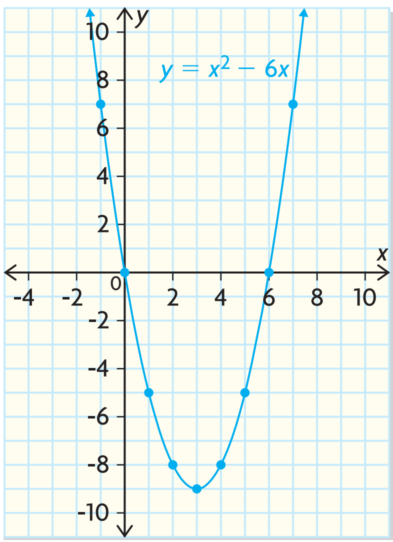
Sketch the graph of the relation $y = x^2 - 6x$. Determine the equation of the axis of symmetry, the coordinates of the vertex, the y-intercept, and the x-intercepts.

Cassandra's Solution

The relation $y = x^2 - 6x$ is quadratic.
$a = 1$, $b = -6$, and $c = 0$

The degree of the equation is 2, so the graph is a parabola. The coefficient of the x^2 term is $a = 1$. Since a is positive, the parabola opens upward.

x	y
−1	7
0	0
1	−5
2	−8
3	−9
4	−8
5	−5
6	0
7	7



I created a table of values using some negative and some positive x-values. I plotted each ordered pair, and drew a parabola that passed through each point. The parabola appears to have (3, −9) as its vertex and $x = 3$ as its axis of symmetry.

The points (1, −5) and (5, −5) are directly across from each other on the parabola.

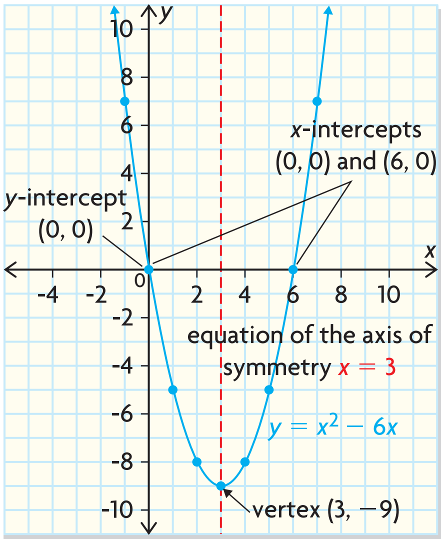
$x = \frac{1 + 5}{2}$ so $x = 3$

The equation of the axis of symmetry is $x = 3$.

I verified the equation of the axis of symmetry by averaging the x-coordinates of two points with the same y-value.

When $x = 3$ in $y = x^2 - 6x$,
$y = 3^2 - 6(3)$
$y = 9 - 18$
$y = -9$
The vertex occurs at (3, −9).

Since the vertex is on the axis of symmetry and the parabola, I substituted $x = 3$ into $y = x^2 - 6x$.



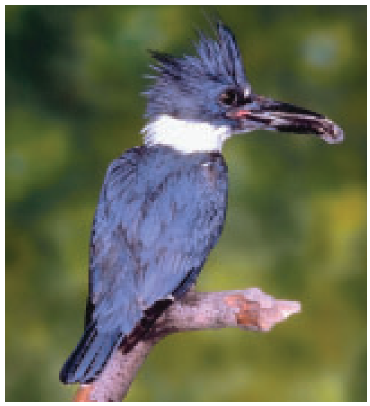
I looked at my graph to determine its features.

This parabola has $x = 3$ as the equation of its axis of symmetry, the vertex is located at $(3, -9)$, the y-intercept is 0, and the x-intercepts are 0 and 6.

EXAMPLE 3 Selecting a strategy to determine the minimum value

A kingfisher dives into a lake. The underwater path of the bird is described by a parabola with the equation $y = 0.5x^2 - 3x$, where x is the horizontal position of the bird relative to its entry point and y is the depth of the bird underwater. Both measurements are in metres.

Graph the parabola. Use your graph to determine the equation of the axis of symmetry, the coordinates of the vertex, the y-intercept, and the x-intercepts. Calculate the bird's greatest depth below the water surface.



Environment *Connection*

Since the Belted Kingfisher eats fish and crayfish, it is at risk due to toxins such as mercury.

Pauline's Solution

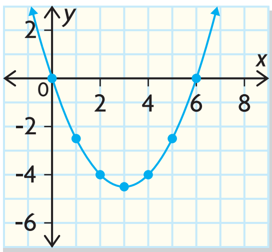
Since $a = 0.5$, the parabola opens upward. The deepest point of the kingfisher's path is the **minimum value** of the relation. This occurs at the vertex of the parabola and corresponds to the y-coordinate of the vertex.

I made a plan to solve the problem.

minimum value
the least value of the dependent variable in a relation

x	*y*
0	0.0
1	−2.5
2	−4.0
3	−4.5
4	−4.0
5	−2.5
6	0.0

I created a table of values using the equation. I assumed that the bird moved from left to right, so I chose only positive values of *x*.



I plotted the points to create a graph; the vertex of the parabola appears to be (3, −4.5) and the equation of the axis of symmetry appears to be $x = 3$.

The y-intercept occurs at (0, 0).
The x-intercepts occur at (0, 0) and (6, 0).

I looked at my graph to determine the intercepts.

$$x = \frac{0 + 6}{2}$$

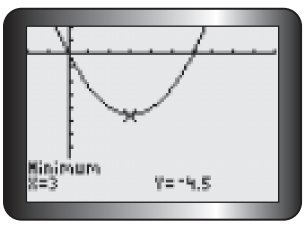
The equation of the axis of symmetry is $x = 3$.

The axis of symmetry is halfway between the x-intercepts, so I calculated the mean of the x-coordinates.

$$y = 0.5x^2 - 3x$$
$$y = 0.5(3)^2 - 3(3)$$
$$y = 4.5 - 9$$
$$y = -4.5$$

The vertex is on the axis of symmetry, so I substituted $x = 3$ into the equation to determine the y-coordinate.

The vertex is (3, −4.5).
The greatest depth of the bird, below the surface of the water, is 4.5 m.



I verified the minimum value of the relation using the minimum operation on a graphing calculator.

Tech *Support*

For help graphing a relation and determining its minimum value using a TI-83/84 graphing calculator, see Appendix B-9. If you are using a TI-*n*spire, see Appendix B-45.

EXAMPLE 4 | Connecting a situation to a quadratic model

A model rocket is shot into the air from the roof of a building. Its height, h, above the ground, measured in metres, can be modelled by the equation $h = -5t^2 + 35t + 5$, where t is the time elapsed since liftoff in seconds.

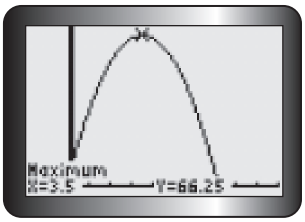
a) Determine the greatest height reached by the rocket.
b) How long is the rocket in flight?
c) Determine the height of the building.
d) When is the height of the rocket 61.25 m?

Liam's Solution

a) The equation is quadratic. Since $a = -5$, the graph is a parabola that opens downward. The greatest height occurs at the vertex. I entered the equation $y = -5x^2 + 35x + 5$ into a graphing calculator.

Since the calculator uses the variables x and y, I replaced the dependent variable h with y and the independent variable t with x.

I adjusted the window settings until I could see the vertex. I used the maximum operation to determine the coordinates.



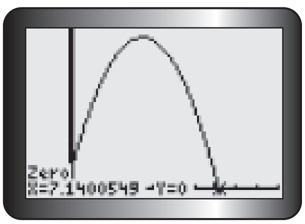
The vertex is (3.5, 66.25).

At 3.5 s after liftoff, the rocket reaches its greatest height of 66.25 m.

Tech | *Support*

For help graphing a relation, determining its maximum value, and determining its x-intercepts using a TI-83/84 graphing calculator, see Appendix B-2, B-9, and B-8. If you are using a TI-*n*spire, see Appendix B-38, B-45, and B-44.

b)



The second zero (or x-intercept) corresponds to the rocket hitting the ground. I used the zero operation to determine the coordinates of this point.

The rocket is in flight for about 7.14 s.

Communication | *Tip*

The zeros of a relation are its x-intercepts. "Zero" is another name for "x-intercept."

c) Let $x = 0$.

$y = -5(0)^2 + 35(0) + 5$

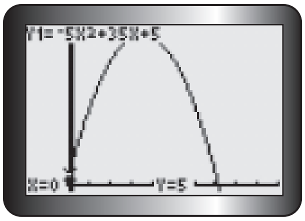
$y = 0 + 0 + 5$

$y = 5$

The height of the building corresponds to the *y*-intercept of the graph. This is the initial height of the ball, so I substituted $x = 0$ into the equation and solved for *y*.

Tech | ***Support***

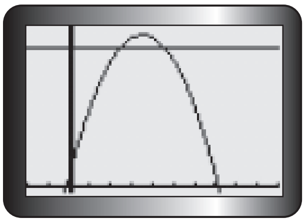
For help determining the value of a relation using a TI-83/84 graphing calculator, see Appendix B-3. If you are using a TI-*n*spire, see Appendix B-39.



I verified my answer on a graphing calculator, using the value operation.

The building is 5.00 m tall.

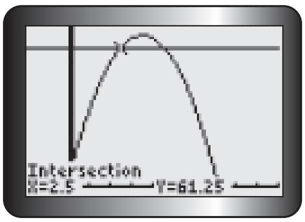
d)

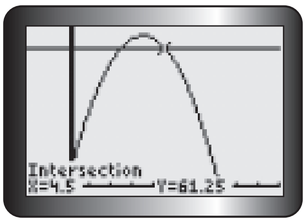


I had to determine when the height is 61.25 m. I entered the relation $y = 61.25$ into Y2 of the equation editor and re-graphed. The *x*-coordinates of the points of intersection of the horizontal line and the parabola tell me when this height occurs.

Tech | ***Support***

For help determining the points of intersection for two relations using a TI-83/84 graphing calculator, see Appendix B-11. If you are using a TI-*n*spire, see Appendix B-47.





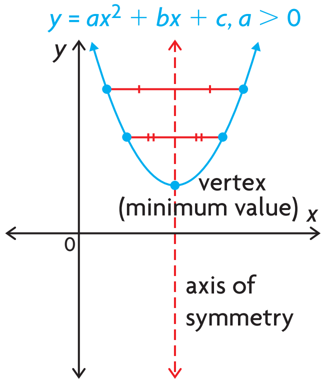
I used the intersect operation to determine the points of intersection. The first coordinate of each point represents a time when the rocket is 61.25 m above the ground.

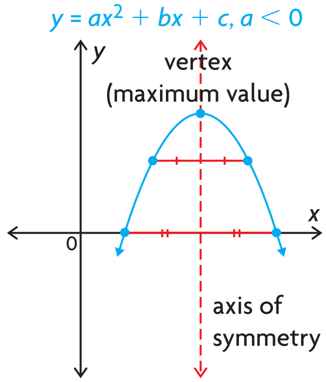
The rocket reaches a height of 61.25 m after 2.5 s on the way up and after 4.5 s on the way down.

In Summary

Key Ideas

- The vertex of a parabola with equation $y = ax^2 + bx + c$ is the point on the graph with
  - the least y-coordinate, or minimum value, if the parabola opens upward
  - the greatest y-coordinate, or maximum value, if the parabola opens downward





- A parabola with equation $y = ax^2 + bx + c$ is symmetrical with respect to a vertical line through its vertex. This line, or axis of symmetry, is the perpendicular bisector of any line segment that joins two points with the same y-coordinate on the parabola.

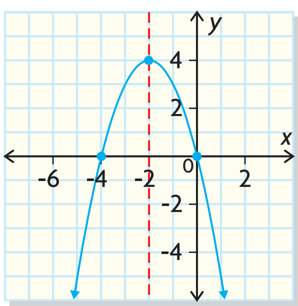
Need to Know

- The x-intercepts, or zeros, of a parabola can be determined by setting $y = 0$ in the equation of the parabola and solving for x.
- The y-intercept of a parabola can be determined by setting $x = 0$ in the equation of the parabola and solving for y.
- When a problem can be modelled by a quadratic relation, the graph of the relation can be used to estimate solutions to the problem.

CHECK Your Understanding

1. For each graph, state the y-intercept, the zeros, the coordinates of the vertex, and the equation of the axis of symmetry.

a)



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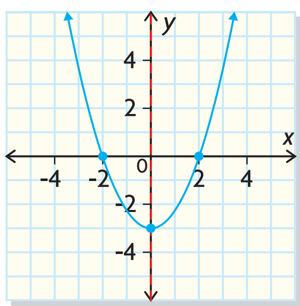

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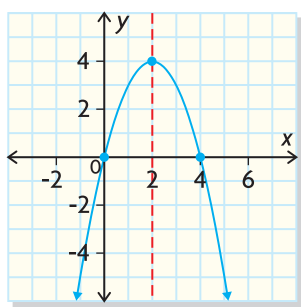


2. State the maximum or minimum value of each relation in question 1.

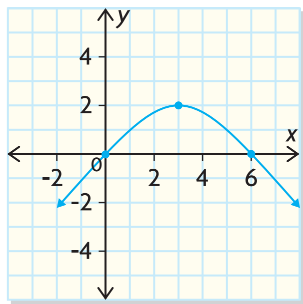
3. Two parabolas have the same x-intercepts, at (0, 0) and (10, 0). One parabola has a maximum value of 2. The other parabola has a minimum value of -4. Sketch the graphs of the parabolas on the same axes.

PRACTISING

4. Examine each parabola.

i) Determine the coordinates of the vertex.
ii) Determine the zeros.
iii) Determine the equation of the axis of symmetry.
iv) If you calculated the second differences, would they be positive or negative? Explain.

a)



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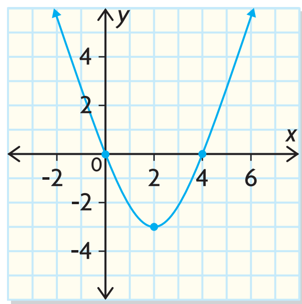


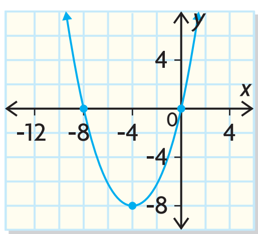
5. The zeros of a quadratic relation occur at $x = 0$ and $x = 6$. The second differences are positive.

a) Is the y-value of the vertex a maximum value or a minimum value? Explain.
b) Is the y-value of the vertex a positive number or a negative number? Explain.
c) Determine the x-value of the vertex.

6. For each quadratic relation, state

i) the equation of the axis of symmetry
ii) the coordinates of the vertex
iii) the y-intercept
iv) the zeros
v) the maximum or minimum value

a)



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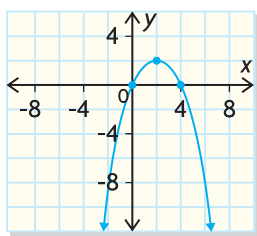
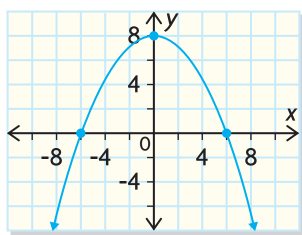


c)



7. Create a table of values for each quadratic relation, and sketch its graph. Then determine

i) the equation of the axis of symmetry
ii) the coordinates of the vertex
iii) the y-intercept
iv) the zeros
v) the maximum or minimum value

a) $y = x^2 + 2$
b) $y = -x^2 - 1$
c) $y = x^2 - 2x$
d) $y = -x^2 + 4x$
e) $y = x^2 - 2x + 1$
f) $y = -x^2 - 2x + 3$

8. Use technology to graph each quadratic relation below. Then determine

i) the equation of the axis of symmetry
ii) the coordinates of the vertex
iii) the y-intercept
iv) the zeros
v) the maximum or minimum value

a) $y = x^2 - 4x + 3$
b) $y = -x^2 + 4$
c) $y = x^2 + 6x + 8$
d) $y = -x^2 + 6x - 5$
e) $y = 2x(x - 4)$
f) $y = -0.5x(x - 8)$

9. Each pair of points is located on opposite sides of the same parabola. Determine the equation of the axis of symmetry for each parabola.

a) $(3, 2), (9, 2)$
b) $(-18, 3), (7, 3)$
c) $(-5.25, -2.5), (3.75, -2.5)$
d) $\left(-4\frac{1}{2}, 5\right), \left(-1\frac{1}{2}, 5\right)$

10. K Jen knows that $(-1, 41)$ and $(5, 41)$ lie on a parabola defined by the equation $y = 4x^2 - 16x + 21$. What are the coordinates of the vertex?

11. C State whether you agree or disagree with each statement. Explain why.

a) All quadratic relations of the form $y = ax^2 + bx + c$ have two zeros.
b) All quadratic relations of the form $y = ax^2 + bx + c$ have one y-intercept.
c) All parabolas that open downward have second differences that are positive.

Use a graphing calculator to answer questions 12 to 15.

12. A football is kicked into the air. Its height above the ground is approximated by the relation $h = 20t - 5t^2$, where h is the height in metres and t is the time in seconds since the football was kicked.

a) What are the zeros of the relation? When does the football hit the ground?
b) What are the coordinates of the vertex?
c) Use the information you found for parts a) and b) to graph the relation.
d) What is the maximum height reached by the football? After how many seconds does the maximum height occur?

13. A company that manufactures MP3 players uses the relation
A $P = 120x - 60x^2$ to model its profit. The variable x represents the number of thousands of MP3 players sold. The variable P represents the profit in thousands of dollars.

a) What is the maximum profit the company can earn?

b) How many MP3 players must be sold to earn this profit?

c) The company "breaks even" when the profit is zero. Are there any break-even points for this company? If so, how many MP3 players are sold at the break-even points?

Career Connection

Coast guard rescuers drop rafts that inflate within seconds to keep people afloat.

14. An inflatable raft is dropped from a hovering helicopter to a boat in distress below. The height of the raft above the water, in metres, is approximated by the equation $y = 500 - 5x^2$, where x is the time in seconds since the raft was dropped.

a) What is the height of the helicopter above the water?

b) When does the raft reach the water?

c) What is the height of the raft above the water 6 s after it is dropped?

d) When is the raft 100 m above the water?

15. Gamez Inc. makes handheld video game players. Last year,
T accountants modelled the company's profit using the equation $P = -5x^2 + 60x - 135$. This year, accountants used the equation $P = -7x^2 + 70x - 63$. In both equations, P is the profit, in hundreds of thousands of dollars, and x is the number of game players sold, in hundreds of thousands. If the same number of game players were sold in these years, did Gamez Inc.'s profit increase? Justify your answer.

16. a) Explain how the value of a in a quadratic relation, given in standard form, can be used to determine if the quadratic relation has a maximum value or a minimum value.

b) Explain how the coordinates of the vertex are related to the maximum or minimum value of the parabola.

Extending

17. a) Determine first and second differences for each relation.

i) $x = 2y^2$ **iii)** $y = x^3$

ii) $y = 2^x$ **iv)** $y = 2x^4$

b) Are graphs for any of the relations in part a) parabolas? Explain.

c) Are any of the relations in part a) quadratic? Explain.

18. The x-coordinate of the vertex of the graph of $y = 5x^2 - 3.2x + 8$ is $x = 0.32$. The number 0.32 is very similar to -3.2, which is the coefficient of x in the equation. Is this just a coincidence? Investigate several examples. Then make a conjecture and try to prove it.