5.3

Graphing Quadratics in Vertex Form

GOAL

Graph a quadratic relation in the form $y = a(x - h)^2 + k$ by using transformations.

YOU WILL NEED

- grid paper
- ruler

LEARN ABOUT the Math

Srinithi and Kevin are trying to sketch the graph of the quadratic relation $y = 2(x - 3)^2 - 8$ by hand. They know that they need to apply a series of transformations to the graph of $y = x^2$.

? How do you apply transformations to the quadratic relation $y = x^2$ to sketch the graph of $y = 2(x - 3)^2 - 8$?

EXAMPLE 1 Selecting a transformation strategy to graph a quadratic relation

Use transformations to sketch the graph of $y = 2(x - 3)^2 - 8$.

Srinithi's Solution: Applying a horizontal translation first

$y = x^2$

x	−2	−1	0	1	2
y	4	1	0	1	4

I began by graphing $y = x^2$ using five key points. The quadratic relation $y = 2(x - 3)^2 - 8$ is expressed in vertex form.

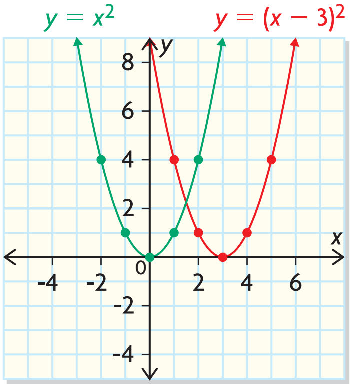
vertex form

a quadratic relation of the form $y = a(x - h)^2 + k$, where the vertex is (h, k)

$y = (x - 3)^2$

x	1	2	3	4	5
y	4	1	0	1	4

Since $h = 3$, I added 3 to the x-coordinate of each point on $y = x^2$. This means that the vertex is (3, 0).

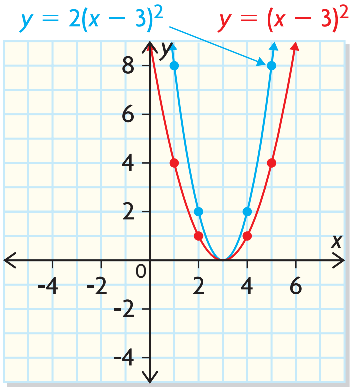


The equation of the new **red** graph is $y = (x - 3)^2$. To draw it, I translated the green parabola 3 units right.

$y = 2(x - 3)^2$

x	1	2	3	4	5
y	8	2	0	2	8

Since $a = 2$, I multiplied all the y-coordinates of the points on the red graph by 2. The vertex stays at (3, 0). The equation of this graph is $y = 2(x - 3)^2$.

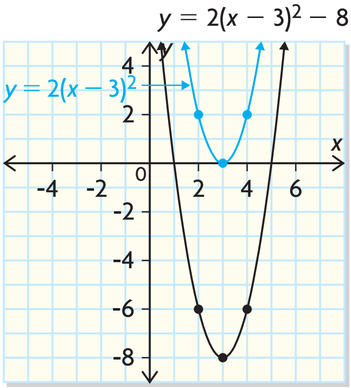


To draw this new **blue** graph, I applied a vertical stretch by a factor of 2 to the red graph. The blue graph looks correct because the graph with the greater a value should be narrower than the other graph.

$y = 2(x - 3)^2 - 8$

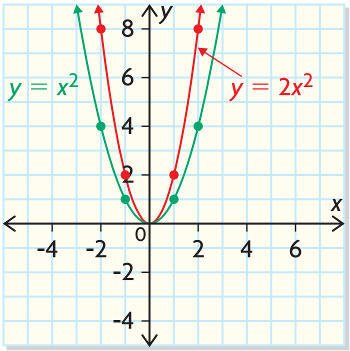
x	1	2	3	4	5
y	0	−6	−8	−6	0

I knew that $k = -8$. I subtracted 8 from the y-coordinate of each point on the blue graph. The vertex is now $(3, -8)$. The equation of the graph is $y = 2(x - 3)^2 - 8$.



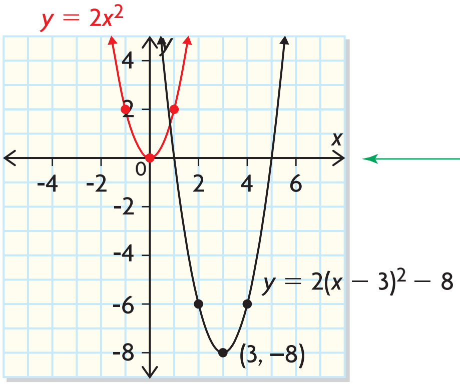
Since $k < 0$, I knew that I had to translate the blue graph 8 units down to get the final **black** graph.

Kevin's Solution: Applying a vertical stretch first



Since $a = 2$, I decided to stretch the graph of $y = x^2$ vertically by a factor of 2. I multiplied the y-coordinate of each point on the graph of $y = x^2$ by 2.

The equation of the resulting **red** graph is $y = 2x^2$.



I applied both translations in one step. Adding 3 to the *x*-coordinate and subtracting 8 from the *y*-coordinate from each point on the red graph causes the red graph to move 3 units right and 8 units down.

The equation of the resulting **black** graph is $y = 2(x - 3)^2 - 8$.

Reflecting

A. Why was it not necessary for Kevin to use two steps for the translations? In other words, why did he not have to shift the graph to the right in one step, and then down in another step?

B. What are the advantages and disadvantages of each solution?

C. How can thinking about the order of operations applied to the coordinates of points on the graph of $y = x^2$ help you apply transformations to draw a new graph?

APPLY the Math

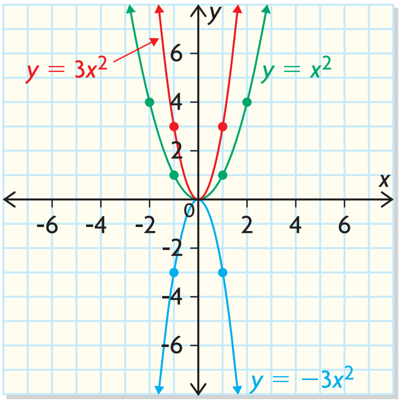
EXAMPLE 2 Reasoning about sketching the graph of a quadratic relation

Sketch the graph of $y = -3(x + 5)^2 + 1$, and explain your reasoning.

Winnie's Solution: Connecting a sequence of transformations to the equation

Applying a vertical stretch of factor 3 and a reflection in the *x*-axis gives the graph of $y = -3x^2$.

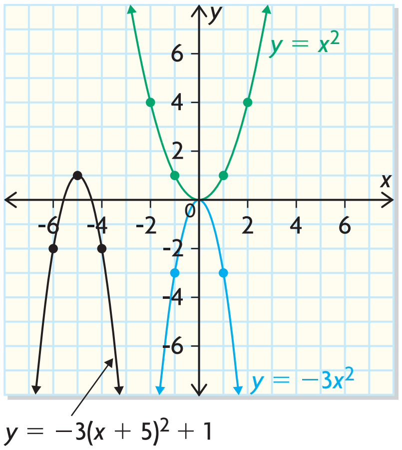
In the quadratic relation $y = -3(x + 5)^2 + 1$, the value of *a* is -3. This represents a vertical stretch by a factor of 3 and a reflection in the *x*-axis.



I noticed that I can combine the stretch and reflection into a single step by multiplying each *y*-coordinate of points on $y = x^2$ by -3.

In the equation, $h = -5$ and $k = 1$. Therefore, the vertex is at $(-5, 1)$. I translated the blue graph 5 units left and 1 unit up.

I determined that the vertex is $(-5, 1)$. Then I shifted all the points on the graph of $y = -3x^2$ so that they were 5 units left and 1 unit up.



I drew a smooth curve through the new points to sketch the graph.

Beth's Solution: Connecting the properties of a parabola to the equation

Based on the equation $y = -3(x + 5)^2 + 1$, the parabola has these properties:

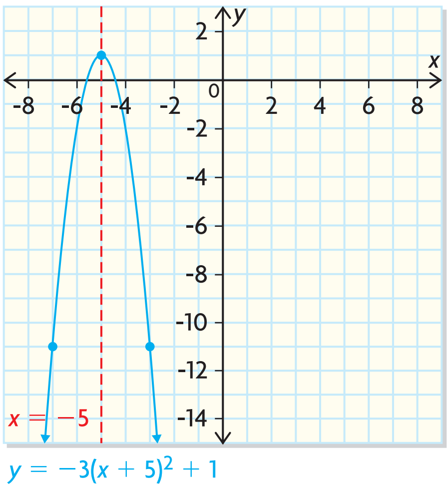
- Since $a < 0$, the parabola opens downward.
- The vertex of the parabola is $(-5, 1)$.
- The equation of the axis of symmetry is $x = -5$.

Since the equation was given in vertex form, I listed the properties of the parabola that I could determine from the equation.

$$y = -3(-3 + 5)^2 + 1$$
$$y = -3(2)^2 + 1$$
$$y = -12 + 1$$
$$y = -11$$

To determine another point on the parabola, I let $x = -3$.

Therefore, $(-3, -11)$ is a point on the parabola.



I plotted the vertex and the point I had determined, $(-3, -11)$. Then I drew the axis of symmetry. I used symmetry to determine the point directly across from $(-3, -11)$. This point is $(-7, -11)$.

I plotted the points and joined them with a smooth curve.

EXAMPLE 3 **Reasoning** about the effects of transformations on a quadratic relation

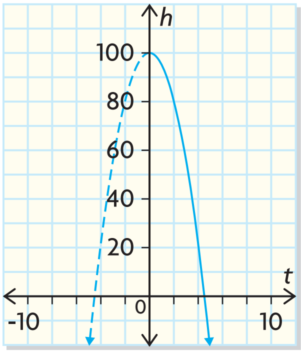
For a high school charity event, the principal pays to drop a watermelon from a height of 100 m. The height, h, in metres, of the watermelon after t seconds is $h = -0.5gt^2 + k$, where g is the acceleration due to gravity and k is the height from which the watermelon is dropped.
On Earth, $g = 9.8$ m/s^2.

a) The clock that times the fall of the watermelon runs for 3 s before the principal releases the watermelon. How does this change the graph shown? Determine the equation of the new relation.

b) On Mars, $g = 3.7$ m/s^2. Suppose that an astronaut dropped a watermelon from a height of 100 m on Mars. Determine the equation for the height of the watermelon on Mars. How does the graph for Mars compare with the graph for Earth for part a)?

c) The principal drops another watermelon from a height of 50 m on Earth. How does the graph for part a) change? How does the relation change?

d) Repeat part c) for an astronaut on Mars.



$h = -4.9t^2 + 100, t \geq 0$

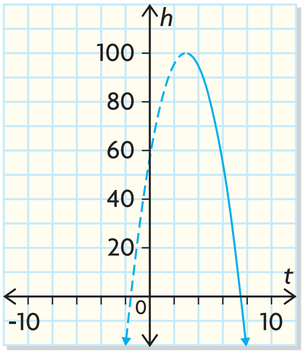
Nadia's Solution

a) The equation of the original relation is
$h = -0.5(9.8)t^2 + 100$
$h = -4.9t^2 + 100$, where $t \geq 0$

The original graph is a parabola that opens downward, with vertex $(0, k) = (0, 100)$. I wrote and simplified the original relation. Only the right branch of the parabola makes sense in this situation since time can't be negative.

The parabola is translated 3 units right.
The equation of the new relation is
$h = -4.9(t - 3)^2 + 100$, where $t \geq 3$.

I subtracted 3 from the t-coordinate to determine the new relation. Since the watermelon is not falling before 3 s, the relation only holds for $t \geq 3$.



$h = -4.9(t - 3)^2 + 100, t \geq 3$

If the clock runs for 3 s before the watermelon is dropped, then the watermelon will be at its highest point at 3 s. So, the vertex of the new parabola is (3, 100), which is a shift of the original parabola 3 units right.

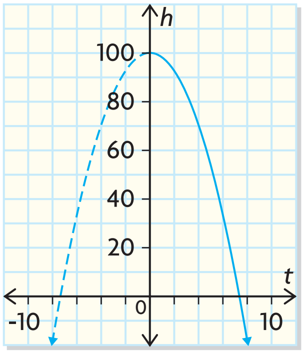
b) The equation of the relation on Mars is

$h = -0.5(3.7)t^2 + 100$

$h = -1.85t^2 + 100$, where $t \geq 0$

I used the value of g on Mars, $g = 3.7\text{m/s}^2$, instead of $g = 9.8\ \text{m/s}^2$.

The graph for Mars is wider near the vertex.



$h = -1.85t^2 + 100,\ t \geq 0$

A lesser (negative) *a*-value means that the parabola is wider.

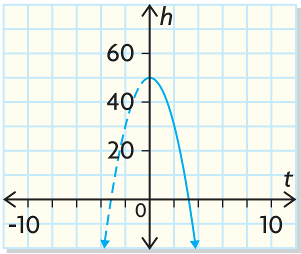
The *t*-intercept is farther from the origin, so the watermelon would take longer to hit the ground on Mars compared to Earth.

c) The equation of the new relation is

$h = -4.9t^2 + 50$, where $t \geq 0$.

In the relation, *k* changes from 100 to 50.

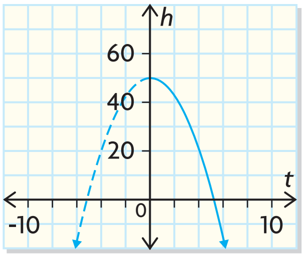
The new graph has the same shape but is translated 50 units down.



$h = -4.9t^2 + 50,\ t \geq 0$

The new vertex is half the distance above the origin, at (0, 50) instead of (0, 100). This is a shift of 50 units down.

d) The new graph for Mars is wider than the original graph and is translated 50 units down.



$h = -1.85t^2 + 50,\ t \geq 0$

The new graph for Mars is wider than the original graph, like the graph for part b). It is translated down, like the graph for part c).

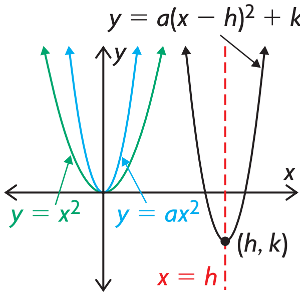
In Summary

Key Idea

- Compared with the graph of $y = x^2$, the graph of $y = a(x - h)^2 + k$ is a parabola that has been stretched or compressed vertically by a factor of a, translated horizontally by h, and translated vertically by k. As well, if $a < 0$, the parabola is reflected in the x-axis.

Need to Know

- The vertex of $y = a(x - h)^2 + k$ has the coordinates (h, k). The equation of the axis of symmetry of $y = a(x - h)^2 + k$ is $x = h$.
- When sketching the graph of $y = a(x - h)^2 + k$ as a transformation of the graph of $y = x^2$, follow the order of operations for the arithmetic operations to be performed on the coordinates of each point. Apply vertical stretches/compressions and reflections, which involve multiplication, before translations, which involve addition or subtraction.



CHECK Your Understanding

1. Describe the transformations you would apply to the graph of $y = x^2$, in the order you would apply them, to obtain the graph of each quadratic relation.

a) $y = x^2 - 3$

b) $y = (x + 5)^2$

c) $y = -\frac{1}{2}x^2$

d) $y = 4(x + 2)^2 - 16$

2. For each quadratic relation in question 1, identify

i) the direction in which the parabola opens

ii) the coordinates of the vertex

iii) the equation of the axis of symmetry

3. Sketch the graph of each quadratic relation. Start with a sketch of $y = x^2$, and then apply the appropriate transformations in the correct order.

a) $y = (x + 5)^2 - 4$

b) $y = -0.5x^2 + 8$

c) $y = 2(x - 3)^2$

d) $y = \frac{1}{2}(x - 4)^2 - 2$

PRACTISING

4. What transformations would you apply to the graph of $y = x^2$ to create the graph of each relation? List the transformations in the order you would apply them.

a) $y = -x^2 + 9$

b) $y = (x - 3)^2$

c) $y = (x + 2)^2 - 1$

d) $y = -x^2 - 6$

e) $y = -2(x - 4)^2 + 16$

f) $y = \frac{1}{2}(x + 6)^2 + 12$

g) $y = -\frac{1}{2}(x + 4)^2 - 7$

h) $y = 5(x - 4)^2 - 12$

5. Sketch a graph of each quadratic relation in question 4 on a separate grid. Use the properties of the parabola and additional points as necessary.

6. Match each equation with the correct graph.

a) $y = \frac{1}{2}(x - 2)^2 - 5$

b) $y = \frac{1}{2}(x - 4)^2 - 2$

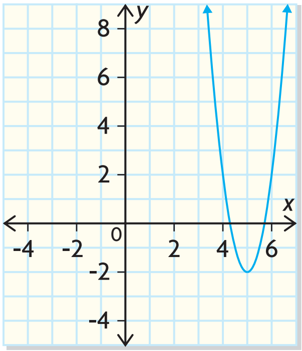
c) $y = -2(x + 2)^2 + 5$

d) $y = -2(x - 2)^2 - 5$

e) $y = 4(x - 5)^2 - 2$

f) $y = \frac{1}{3}x^2 - 2$

i)



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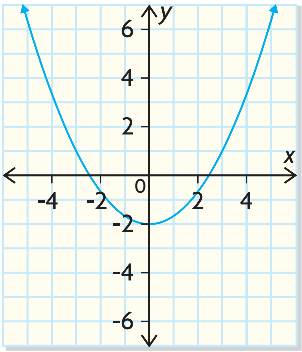

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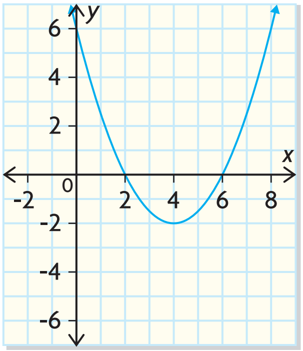

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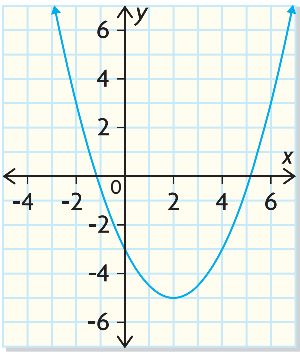

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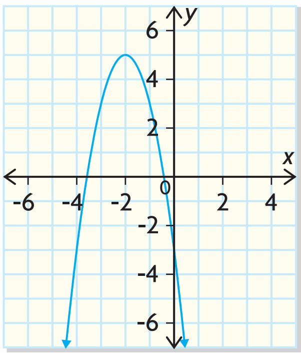

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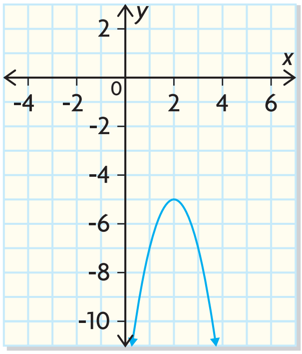


7. Sketch the graph of each quadratic relation by hand. Start with a sketch of $y = x^2$, and then apply the appropriate transformations in the correct order.

a) $y = -(x - 2)^2$

b) $y = \frac{1}{2}(x + 2)^2 - 8$

c) $y = -3(x - 1)^2 + 7$

d) $y = \frac{3}{4}x^2 - 5$

e) $y = \frac{1}{2}(x - 2)^2 - 5$

f) $y = -1.5(x + 3)^2 + 10$

8. Copy and complete the following table.

Quadratic Relation	Stretch/ Compression Factor	Reflection in the x-axis	Horizontal/ Vertical Translation	Vertex	Axis of Symmetry
	3	no	right 2, down 5	$(2, -5)$	$x = 2$
$y = 4(x + 2)^2 - 3$					
$y = -(x - 1)^2 + 4$					
$y = 0.8(x - 6)^2$					
$y = 2x^2 - 5$					

9. **C** Determine the equations of three different parabolas with a vertex at $(-2, 3)$. Describe how the graphs of the parabolas are different from each other. Then sketch the graphs of the three relations on the same set of axes.

10. When an object with a parachute is released to fall freely, its height, h, in metres, after t seconds is modelled by $h = -0.5(g - r)t^2 + k$, where g is the acceleration due to gravity, r is the resistance offered by the parachute, and k is the height from which the object is dropped. On Earth, $g = 9.8$ m/s^2. The resistance offered by a single bed sheet is 0.6 m/s^2, by a car tarp is 2.1 m/s^2, and by a regular parachute is 8.9 m/s^2.

a) Describe how the graphs will differ for objects dropped from a height of 100 m using each of the three types of parachutes.

b) Is it possible to drop an object attached to the bed sheet and a similar object attached to a regular parachute and have them hit the ground at the same time? Describe how it would be possible and what the graphs of each might look like.

11. Write the equation of a parabola that matches each description.

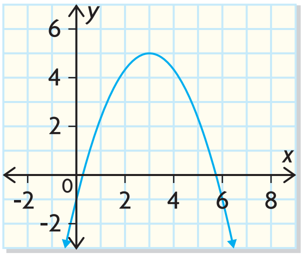
a) The graph of $y = x^2$ is reflected about the x-axis and then translated 5 units up.

b) The graph of $y = x^2$ is stretched vertically by a factor of 5 and then translated 2 units left.

c) The graph of $y = x^2$ is compressed vertically by a factor of $\frac{1}{5}$ and then translated 6 units down.

d) The graph of $y = x^2$ is reflected about the x-axis, stretched vertically by a factor of 6, translated 3 units right, and translated 4 units up.

12. **K** Sketch the graph of each parabola described in question 11 by applying the given sequence of transformations. Use a separate grid for each graph.



13. Which equation represents the graph shown at the left? Explain your reasoning.

a) $y = -\frac{2}{3}x^2 + 5$

b) $y = -(x - 3)^2 + 5$

c) $y = -\frac{2}{3}(x - 3)^2 + 5$

d) $y = \frac{2}{3}(x - 3)^2 + 5$

Safety Connection

A helmet and goggles are important safety equipment for skydivers.

14. A sky diver jumped from an airplane. He used his watch to time the length of his jump. His height above the ground can be modelled by $h = -5(t - 4)^2 + 2500$, where h is his height above the ground in metres and t is the time in seconds from the time he started the timer.

a) How long did the sky diver delay his jump?

b) From what height did he jump?

15. **A** A video tracking device recorded the height, h, in metres, of a baseball after it was hit. The data collected can be modelled by the relation $h = -5(t - 2)^2 + 21$, where t is the time in seconds after the ball was hit.

a) Sketch a graph that represents the height of the baseball.

b) What was the maximum height reached by the baseball?

c) When did the baseball reach its maximum height?

d) At what time(s) was the baseball at a height of 10 m?

e) Approximately when did the baseball hit the ground?

16. **T** When a graph of $y = x^2$ is transformed, the point (3, 9) moves to (8, 17). Describe three sets of transformations that could make this happen. For each set, give the equation of the new parabola.

17. Express the quadratic relation $y = 2(x - 4)(x + 10)$ in both standard form and vertex form.

18. Copy and complete the chart to show what you know about the quadratic relation $y = -2(x + 3)^2 + 4$.

Translation:	Reflection:
Stretch/ Compression:	Vertex:

$y = -2(x + 3)^2 + 4$

Extending

19. Determine one of the zeros of the quadratic relation

$$y = \left(x - \frac{k}{2}\right)^2 - \frac{(k - 2)^2}{4}.$$