4.4 Factoring Quadratics: $ax^2 + bx + c$

GOAL

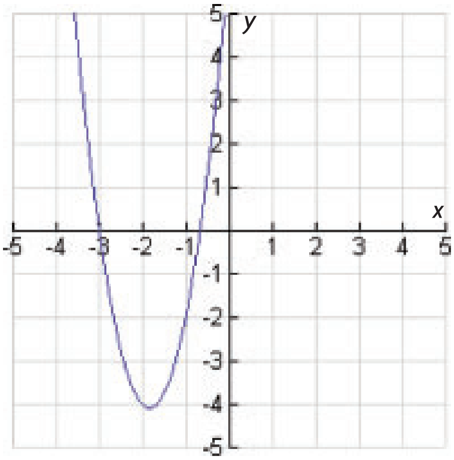
Factor quadratic expressions of the form $ax^2 + bx + c$, where $a \neq 1$.

YOU WILL NEED

- algebra tiles

LEARN ABOUT the Math

Kellie was asked to determine the x-intercepts of $y = 3x^2 + 11x + 6$ algebraically. She created a graph using graphing technology and estimated that the x-intercepts are about $x = -0.6$ and $x = -3$.



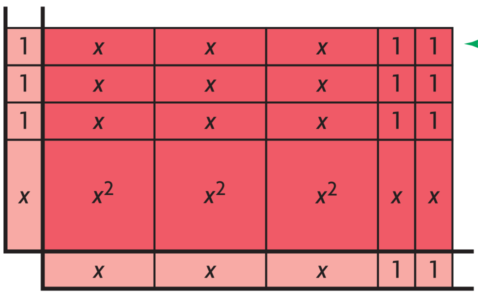
Kellie knows that if she can write the equation in factored form, she can use the factors to determine the x-intercepts. She is unsure about how to proceed because the first term in the expression has a coefficient of 3 and there is no common factor.

? How can you factor $3x^2 + 11x + 6$?

EXAMPLE 1 **Selecting a strategy** to factor a trinomial, where $a \neq 1$

Factor $3x^2 + 11x + 6$, and determine the x-intercepts of $y = 3x^2 + 11x + 6$.

Ellen's Solution: Selecting an algebra tile model



$3x^2 + 11x + 6 = (3x + 2)(x + 3)$

I used tiles to create a rectangular area model of the trinomial.

I placed the tiles along the length and width to read off the factors. The length is $3x + 2$, and the width is $x + 3$.

The equation in factored form is
$y = (3x + 2)(x + 3)$.
Let $3x + 2 = 0$ or $x + 3 = 0$.

$$3x = -2 \qquad x = -3$$
$$x = -\frac{2}{3}$$

The x-intercepts occur when $y = 0$. This happens when either factor is equal to 0.

The x-intercepts are $-\frac{2}{3}$ and -3.

Neil's Solution: Selecting an area diagram and a systematic approach

	qx	$+\ s$
px	pqx^2	psx
$+\ r$	qrx	rs

I thought about the general situation, where $(px + r)$ and $(qx + s)$ represent the unknown factors. I created an area model and used it to look for patterns between the coefficients in the factors and the coefficients in the trinomial.

$$\begin{aligned}(px + r)(qx + s) &= pqx^2 + psx + qrx + rs \\ &= pqx^2 + (ps + qr)x + rs\end{aligned}$$

Suppose that
$3x^2 + 11x + 6 = (px + r)(qx + s)$.

I imagined writing two factors for this product. I had to figure out the coefficients and the constants in the factors.

$$\begin{aligned}(px + r)(qx + s) &= pqx^2 + (ps + qr)x + rs \\ &= 3x^2 + 11x + 6\end{aligned}$$

I matched the coefficients and the constants.

p	q	r	s	$ps + qr$	
3	1	3	2	9	
1	3	2	3	9	
1	3	3	2	11	√

I needed values of p and q that, when multiplied, would give a product of 3. I also needed values of r and s that would give a product of 6.

$p = 1$, $q = 3$, $r = 3$, and $s = 2$
$3x^2 + 11x + 6 = (x + 3)(3x + 2)$

The middle coefficient is 11, so I tried different combinations of $ps + qr$ to get 11.

The equation in factored form is
$y = (x + 3)(3x + 2)$.
Let $x + 3 = 0$ or $3x + 2 = 0$.
$x = -3 \qquad 3x = -2$
$x = -\frac{2}{3}$

The x-intercepts are -3 and $-\frac{2}{3}$.

The x-intercepts occur when $y = 0$. This happens when either factor is equal to zero.

Astrid's Solution: Selecting a decomposition strategy

$3x^2 + 11x + 6$
$= (px + r)(qx + s)$

I imagined writing two factors for this product. I had to figure out the coefficients and the constants in the factors.

$(px + r)(qx + s)$
$= pxqx + pxs + rqx + rs$
$= pqx^2 + (qr + ps)x + rs$

I multiplied the binomials. I noticed that I would get the product of all four missing values if I multiplied the coefficient of x^2 (pq) and the constant (rs).

ps and qr, the two values that are added to get the coefficient of the middle term, are both factors of $pqrs$.

$3x^2 + 11x + 6$
$= 3x^2 + ?x + ?x + 6$

If I added the product of two of these values (ps) to the product of the other two (qr), I would get the coefficient of x.

$3 \times 6 = 18$
The factors of 18 are 1, 2, 3, 6, 9, and 18.

$11 = 9 + 2$

I needed to **decompose** the 11 from $11x$ into two parts. Each part had to be a factor of 18, because $3 \times 6 = 18$.

decompose
break a number or an expression into the parts that make it up

$3x^2 + 9x + 2x + 6$
$= \underline{3x^2 + 9x} + \underline{2x + 6}$
$= 3x(x + 3) + 2(x + 3)$

$= (x + 3)(3x + 2)$

I divided out the greatest common factors from the first two terms and then from the last two terms.

I factored out the binomial common factor.

The equation in factored form is
$y = (x + 3)(3x + 2)$.
Let $x + 3 = 0$ or $3x + 2 = 0$.

$$x = -3 \qquad 3x = -2$$
$$x = -\frac{2}{3}$$

The x-intercepts are -3 and $-\frac{2}{3}$.

The x-intercepts occur when $y = 0$. This happens when either factor is equal to zero.

Reflecting

A. Explain how Ellen's algebra tile arrangement shows the factors of the expression.

B. How is Neil's strategy similar to the strategy used to factor trinomials of the form $x^2 + bx + c$? How is it different?

C. How would Astrid's decomposition change if she had been factoring $3x^2 + 22x + 24$ instead?

D. Which factoring strategy do you prefer? Explain why.

APPLY the Math

EXAMPLE 2 **Selecting a systematic strategy** to factor a trinomial, where $a \neq 1$

Factor $4x^2 - 8x - 5$.

Katie's Solution

$$4x^2 - 8x - 5 = (px + r)(qx + s)$$
$$= pqx^2 + (ps + qr)x + rs$$
$$pq = 4 \quad \text{and} \quad rs = -5$$

p	q
1	4
4	1
2	2

r	s
1	−5
−5	1

I wrote the quadratic as the product of two binomials with unknown coefficients and constants. Then I listed all the possible pairs of values for pq and rs.

$$pqx^2 + (ps + qr)x + rs = 4x^2 - 8x - 5$$
$$ps + qr = -8$$

I had to choose values that would make $ps + qr = -8$.

$(px + r)(qx + s) = (2x + 1)(2x - 5)$

So, $4x^2 - 8x - 5 = (2x + 1)(2x - 5)$.

The values $p = 2$, $q = 2$, $r = 1$, and $s = -5$ work because
- pq is $(2)(2) = 4$
- rs is $(1)(-5) = -5$
- $ps + qr$ is $(2)(-5) + (2)(1) = -8$

$$\begin{aligned}(2x + 1)(2x - 5) &= 4x^2 - 10x + 2x - 5\\ &= 4x^2 - 8x - 5\end{aligned}$$

I checked by multiplying.

EXAMPLE 3 Selecting a decomposition strategy to factor a trinomial

Factor $12x^2 - 25x + 12$.

Braedon's Solution

$$\begin{aligned}&12x^2 - 25x + 12\\ &= 12x^2 - 16x - 9x + 12\end{aligned}$$

I looked for two numbers whose sum is -25 and whose product is $(12)(12) = 144$. I knew that both numbers must be negative, since the sum is negative and the product is positive. The numbers are -16 and -9. I used these numbers to decompose the middle term.

$$\begin{aligned}&= \underline{12x^2 - 16x} - \underline{9x + 12}\\ &= 4x(3x - 4) - 3(3x - 4)\\ &= (3x - 4)(4x - 3)\end{aligned}$$

I factored the first two terms and then the last two terms. Then I divided out the common factor of $3x - 4$.

EXAMPLE 4 Selecting a guess-and-test strategy to factor a trinomial

Factor $7x^2 + 19x - 6$.

Dylan's Solution

$7x^2 + 19x - 6$

	$7x$	?
x	$7x^2$	?
?	?	-6

I thought of the product of the factors as the dimensions of a rectangle with the area $7x^2 + 19x - 6$.

The only factors of $7x^2$ are $7x$ and x. The factors of -6 are -6 and 1, -2 and 3, 6 and -1, and 2 and -3. I had to determine which factors of $7x^2$ and -6 would add to $19x$.

	$7x$	-6
x	$7x^2$	$-6x$
1	$7x$	-6

I used trial and error to determine the values in place of the question marks. Then I checked by multiplying.

$(7x - 6)(x + 1) = 7x^2 + x - 6$ wrong factors

	$7x$	-2
x	$7x^2$	$-2x$
3	$21x$	-6

I repeated this process until I found the combination that worked.

$(7x - 2)(x + 3) = 7x^2 + 19x - 6$ worked

$7x^2 + 19x - 6 = (7x - 2)(x + 3)$

In Summary

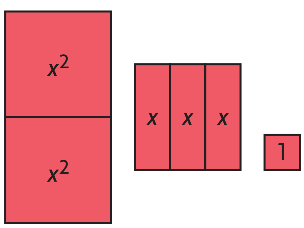
Key Idea

- If the quadratic expression $ax^2 + bx + c$ (where $a \neq 1$) can be factored, then the factors have the form $(px + r)(qx + s)$, where $pq = a$, $rs = c$, and $ps + rq = b$.

Need to Know

- If the quadratic expression $ax^2 + bx + c$ (where $a \neq 1$) can be factored, then the factors can be found by
  - forming a rectangle using algebra tiles
  - using the algebraic model $(px + r)(qx + s) = pqx^2 + (ps + qr)x + rs$ systematically
  - using decomposition
  - using guess and test
- A trinomial of the form $ax^2 + bx + c$ (where $a \neq 1$) can be factored if there are two integers whose product is ac and whose sum is b.

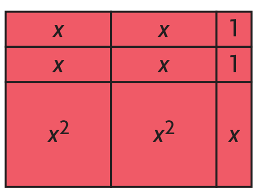
CHECK Your Understanding



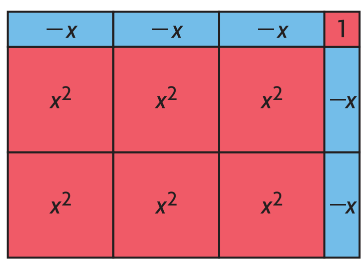
1. a) Write the trinomial that is represented by the algebra tiles at the left.
b) Sketch what the tiles would look like if they were arranged in a rectangle.
c) Use your sketch to determine the factors of the trinomial.

2. Each of the following four diagrams represents a trinomial. Identify the trinomial and its factors.

a)



b)



c)

	?	?
?	$8x^2$	$6x$
?	$20x$	15

d)

	?	?
?	$15x^2$	$-5x$
?	$18x$	-6

3. Determine the missing factor.

a) $2c^2 + 7c - 4 = (c + 4)(\blacksquare)$

b) $4z^2 - 9z - 9 = (\blacksquare)(z - 3)$

c) $6y^2 - y - 1 = (3y + 1)(\blacksquare)$

d) $6p^2 + 7p - 3 = (\blacksquare)(2p + 3)$

PRACTISING

4. Determine the value of each symbol.

a) $5x^2 + \blacklozenge x + 3 = (x + 3)(5x + \blacksquare)$

b) $2x^2 - \blacksquare x - \blacklozenge = (2x + 3)(x - 2)$

c) $12x^2 - 7x + \blacksquare = (3x - \blacklozenge)(4x - \blacklozenge)$

d) $14x^2 - 29x + \blacklozenge = (2x - 3)(7x - \blacksquare)$

5. Factor each expression.

a) $2x^2 + x - 6$

b) $3n^2 - 11n - 4$

c) $10a^2 + 3a - 1$

d) $4x^2 - 16x + 15$

e) $2c^2 + 5c - 12$

f) $6x^2 + 5x + 1$

6. Factor.

a) $6x^2 - 13x + 6$

b) $10m^2 + m - 3$

c) $2a^2 - 11a + 12$

d) $4x^2 - 20x + 25$

e) $5d^2 + 8 - 14d$

f) $6n^2 - 20 + 26n$

7. Factor.

a) $15x^2 + 4x - 4$

b) $18m^2 - 3m - 10$

c) $16a^2 - 50a + 36$

d) $35x^2 - 27x - 18$

e) $63n^2 + 126n + 48$

f) $24d^2 + 35 - 62d$

8. Write three different quadratic trinomials of the form $ax^2 + bx + c$, where $a \neq 1$, that have $(3x - 4)$ as a factor.

9. The area of a rectangle is given by each of the following trinomials.
K Determine expressions for the length and width of the rectangle.

a) $A = 6x^2 + 17x - 3$

b) $A = 8x^2 - 26x + 15$

10. Identify possible integers, k, that allow each quadratic trinomial
T to be factored.

a) $kx^2 + 5x + 2$ **b)** $9x^2 + kx - 5$ **c)** $12x^2 - 20x + k$

11. Factor each expression.

a) $6x^2 + 34x - 12$
b) $18v^2 + 33v - 30$
c) $48c^2 - 160c + 100$
d) $5b^3 - 17b^2 + 6b$
e) $-6x - 51xy + 27xy^2$
f) $-7a^2 - 29a + 30$

12. Determine whether each polynomial has $(k + 5)$ as one of its factors.

a) $k^2 + 9k - 52$
b) $4k^3 + 32k^2 + 60k$
c) $6k^2 + 23k + 7$
d) $10 + 19k - 15k^2$
e) $7k^2 + 29k - 30$
f) $10k^2 + 65k + 75$

13. Examine each quadratic relation below.

i) Express the relation in factored form.
ii) Determine the zeros.
iii) Determine the coordinates of the vertex.
iv) Sketch the graph of the relation.

a) $y = 2x^2 - 9x + 4$ **b)** $y = -2x^2 + 7x + 15$

14. A computer software company models the profit on its latest video
A game using the relation $P = -4x^2 + 20x - 9$, where x is the number of games produced in hundred thousands and P is the profit in millions of dollars.

a) What are the break-even points for the company?
b) What is the maximum profit that the company can earn?
c) How many games must the company produce to earn the maximum profit?

15. Factor each expression.

a) $8x^2 - 13xy + 5y^2$
b) $5a^2 - 17ab + 6b^2$
c) $-12s^2 - sr + 35r^2$
d) $16c^4 + 64c^2 + 39$
e) $14v^6 - 39v^3 + 27$
f) $c^3d^3 + 2c^2d^2 - 8cd$

16. Create a flow chart that would help you decide which strategy
C you should use to factor a given polynomial.

Extending

17. Factor.

a) $6(a + b)^2 + 11(a + b) + 3$
b) $5(x - y)^2 - 7(x - y) - 6$
c) $8(x + 1)^2 - 14(x + 1) + 3$
d) $12(a - 2)^4 + 52(a - 2)^2 - 40$

18. Can a quadratic expression of the form $ax^2 + bx + c$ always be factored if $b^2 - 4ac$ is a **perfect square**? Explain.