Working with Continuous Random Variables and the Normal Distribution

HL2 – Lesson 94

CRV Review Q#3

3 Consider the function defined by

$$f(x)=\begin{cases} k & \text{for } 0<x<2 \\ 2k & \text{for } 2\le x\le 3 \\ 0 & \text{otherwise} \end{cases}$$

a Determine the value of k given that f is a PDF of a random variable X.

b Sketch the graph of f and use it to find the median of X.

c Find the values of $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

CRV Review Q#4

4 A continuous random variable X has PDF defined by $f(x) = ax + b$ where $0 \le x \le 3$ where a and b are constants.

a Find, in terms of a, the value of b.

b Given that the median of X is 1, determine the values of a and b.

c Find the values of $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

CRV Review Q#5

5 A continuous random variable T has PDF defined by

$f(t) = \frac{6}{t^2}$ where $2 \le t \le 3$

Find the values of

a $E(T)$ **b** $Var(T)$

c median of T **d** mode of T.

CRV Review Q – Ex 25

Example 25

A continuous random variable X has probability density function

$f(x) = \frac{k}{x^2 + x}$ for $1 \le X \le 2$

a Use a GDC to determine the value of k correct to 4 decimal places.

b Hence find the value of $P(1 \le X \le 1.75)$

CRV Review Q – Ex 26

Example 26

A continuous random variable X has PDF defined by

$f(x) = \frac{1 + \cos(x)}{\pi}$ where $0 \le x \le \pi$

a Find an expression for the distribution function F of X, (CDF of X).

b Determine the exact value of $P\left(\frac{\pi}{6} \le X \le \frac{\pi}{2}\right)$

CRV Review Q – Ex 27

Example 27

Data collected over a long period of time suggests that the time T that vehicles have to wait until they can enter the main road of Straightcity has PDF

$$f(t) = \frac{5}{6}\left(1 - \frac{2}{5}t\right) \text{ where } 0 \le t \le 2 \text{ minutes}$$

a Find an expression for the distribution function of T (CDF of T).

b Calculate $P(0.5 \le T \le 1.5)$.

Normal Distribution – Equation and Parameters

The **normal distribution** is the most important continuous theoretical probability distribution.

→ The PDF of the normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \text{ where } x \in \mathbb{R}$$

The values of $f(x)$ depend on two parameters, μ and σ.

→ The random variable X described by the PDF is a **normal variable** that follows a normal distribution with mean μ and variance σ^2. We write $X \sim N(\mu, \sigma^2)$

Normal Distribution - Intro

Investigation – the normal curve

1 Use your GDC to graph the normal curves for each of these normal variables.

a $X \sim N(1, 1)$ **b** $X \sim N(2, 1)$

c $X \sim N(3, 1)$ **d** $X \sim N(4, 1)$

Describe differences and similarities between these normal curves.
How does the value of the parameter μ affect the normal curve?

2 Graph the normal curves associated with these normal variables.

e $X \sim N(0, 1)$ **f** $X \sim N(0, 2)$

g $X \sim N(0, 3)$ **h** $X \sim N(0, 4)$

Describe differences and similarities between these normal curves.
How does the value of the parameter σ affect the normal curve?

3 Explore some other normal curves and write down your conclusions about the effects of the parameters parameters μ and σ on the graphs of the normal variables.

4 The normal curves are defined by just two parameters, the mean and the variance of the distribution. Based on your knowledge of the shape of a normal curve, explain why it is not necessary to include the values of the median and the mode of the distribution.

Probabilities in a Normal Distribution

Calculating probabilities of normal variables

Suppose that $X \sim N(\mu, \sigma^2)$. As X is a continuous random variable, you can calculate the probability that X takes values in an interval $[a, b]$.

→ As in any continuous distribution

- $P(a \le X \le b) = \int_a^b f(x)dx$, where $f(x)$ is the PDF of X

Alternatively, if you consider the cumulative normal distribution function F,

- $P(a \le X \le b) = F(b) - F(a)$

Probabilities in a Normal Distribution – Practice

Exercise10L

1 Given $X \sim N(1, 2^2)$, calculate
- **a** $P(0 < X < 1.5)$
- **b** $P(0 < 0.5)$
- **c** $P(X \ge 3)$

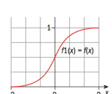
2 If $X \sim N(50, 20^2)$, find the values of
- **a** $P(X < 45)$
- **b** $P(37 \le X < 65)$
- **c** $P(X \ge 52)$

3 Given that $X \sim N(35, 49)$
- **a** state the value of the mean and standard deviation of X
- **b** calculate $P(X < 25)$, $P(29 \le X \le 41)$ and $P(X \ge 45)$

InvNorm

If F is the CDF of a normal variable X, then F is an increasing function and therefore one-to-one.

This means that F has inverse F^{-1} which is useful in determining values of the normal variable X when you are given information about the probability that X is less or greater than certain values.



InvNorm – Ex 32

Example 32

Given $X \sim N(1, 3^2)$ find the values of a and b such that

a $P(X < a) = 0.506$ **b** $P(X > b) = 0.198$

InvNorm - Practice

11. If X is a normal random variable with a mean of 8 and a standard deviation of 1, find the value of c, such that

(a) $p(X > c) = 0.90$ (b) $p(X \le c) = 0.60$

12. If X is a normal random variable with a mean of 50 and a standard deviation of 5, find the value of c, such that

(a) $p(X \le c) = 0.95$ (b) $p(X \ge c) = 0.95$ (c) $p(-c \le X \le c) = 0.95$

Normal Distribution – Investigation #2 - Properties

Investigation – further properties of the normal curve

1 Use your GDC to calculate these probabilities.

- **a** $P(1 < X < 3)$ when $X \sim N(2, 1^2)$
- **b** $P(0 < X < 4)$ when $X \sim N(2, 2^2)$
- **c** $P(-1 < X < 5)$ when $X \sim N(2, 3^2)$
- **d** $P(-1 < X < 3)$ when $X \sim N(2, 2^2)$

What do these examples have in common?
How does it affect the values of the probabilities you calculated?

2 If $X \sim N(\mu, \sigma^2)$ what is the value of $P(\mu - \sigma < X < \mu + \sigma)$?

3 Investigate normal curves further to find the values of

- **a** $P(\mu - 2\sigma < X < \mu + 2\sigma)$
- **b** $P(\mu - 3\sigma < X < \mu + 3\sigma)$
- **c** $P(\mu - 4\sigma < X < \mu + 4\sigma)$

when $X \sim N(\mu, \sigma^2)$

4 Sketch the graph of the normal curve with mean μ and variance σ^2.
If a is positive, use your sketch to compare these probabilities.

- **a** $P(X > \mu + a)$ and $P(X < \mu - a)$
- **b** $P(X > \mu - a)$ and $P(X < \mu + \sigma)$
- **c** $P(X > \mu - a)$ and $P(\mu - \sigma X < \mu + a)$

Normal Distribution Ex 33

Example 33

A variable X follows a normal distribution with mean 2.
Given that $P(X < 3) = 0.8$, find the values of these probabilities.

a $P(X > 3)$
b $P(X < 1)$
c $P(1 < X < 3)$

Properties of the Normal Distribution

Exercise 10N

1 A continuous random variable X follows a normal distribution with mean 5.
Given that $P(X < 3) = 0.3$, and without using a calculator, find these probabilities.
a $P(X \geq 7)$ **b** $P(X < 7)$ **c** $P(3 \leq X < 7)$

2 A continuous random variable Y follows a normal distribution with mean 12.
Given that $P(10 \leq Y < 14) = 0.6$, and without using a calculator, find these probabilities.
a $P(Y \geq 14)$
b $P(Y < 10)$
c $P(12 \leq Y < 14)$
d $P(Y < 14 \mid Y > 12)$

3 If $X \sim N(-5, \sigma^2)$ and $P(X < -3) = 0.8$, write down the values of these probabilities.
a $P(X < -7)$
b $P(-7 \leq X < -5)$
c $P(X < -7) + P(X > -3)$

Standardizing the Normal Distribution

→ All normal curves can be related to a single reference distribution called the **standard normal distribution** which has mean 0 and standard deviation 1.

→ The standard normal variable is denoted by $Z \sim N(0, 1)$. The PDF of Z is always denoted by φ and its CDF by Φ.

Both φ and Φ are very important in statistics and for this reason when you use your GDC to calculate normal probabilities, by default, your calculator assumes that $\mu = 0$ and $\sigma = 1$.

Standardizing the Normal Distribution

Standardized normal variable

If you have to solve a problem where you do not know the mean or the variance of a normal variable you have to use the **standardized normal variable**.

To standardize a random variable $X \sim N(\mu, \sigma^2)$ into the standardized normal variable $Z \sim N(0, 1)$ you use the transformation

$$Z = \frac{x - \mu}{\sigma}$$

which is the algebraic translation of the relation observed between the normal curves of X and Z.

Working with Standardized Normal Distributions

Exercise 100

1. Consider $Z \sim N(0, 1)$. Find, correct to 4 decimal places, the values of:
   - **a** $\Phi(-1.2)$ and $\Phi(1.2)$
   - **b** $\Phi(-2.3)$ and $\Phi(2.3)$
   - **c** $\Phi(-2.6)$ and $\Phi(2.6)$

   Comment on the values obtained.
2. Consider $Z \sim N(0, 1)$. Find the values of
   - **a** $\Phi^{-1}(0.3)$ and $\Phi^{-1}(0.7)$ **b** $\Phi^{-1}(0.4)$ and $\Phi^{-1}(0.6)$

   Comment on the values obtained.
3. Consider $Z \sim N(0, 1)$
   - **a** Find the value of $\varphi(0.45)$ and $\varphi^{-1}(0.45)$
   - **b** Write down the values of a and b such that $P(Z \le a) = 0.5$ and $P(Z \le 0.5) = b$

Working with Standardized Normal Distributions

Example 36

Consider $X \sim N(\mu, 4)$

Find the value of μ given that $P(X \le 2) = 0.556$

Working with Standardized Normal Distributions

Example 37

Consider $X \sim N(\mu, \sigma^2)$
Find the values of μ, and σ given that $P(X \leq 2) = 0.546$ and $P(X \leq 3) = 0.743$