EXAM-STYLE QUESTION

6 Consider the planes defined by the equations $x + y + z = 2$, $2x - y + z = -1$ and $3x - y + kz = 4$, where k is a real number.

a If $k = -3$, find the coordinates of the point of intersection of the three planes.

b Find the value of k for which the three planes do not have any common point.

7 Prove these properties:

a Two planes are either parallel or they intersect in a line.

b A line is either parallel to a plane, intersects it at a single point, or is contained in the plane.

c Two lines perpendicular to the same plane are parallel to each other.

d Two planes perpendicular to the same line are parallel to each other.

Investigation – coefficient patterns

Consider the system $\begin{cases} a_1x + a_2y + a_3z = a_4 \\ a_5x + a_6y + a_7z = a_8 \\ a_9x + a_{10}y + a_{11}z = a_{12} \end{cases}$

Use a GDC to study the solutions of the system when

a $a_1, a_2, \ldots, a_{12}$ are consecutive numbers

b $a_1, a_2, \ldots, a_{12}$ are consecutive even numbers.

In each case, explore the geometric meaning of the pattern found.

11.8 Modeling and problem solving

Extension material on CD: *Worksheet 11*

Vector geometry can be used to model situations that involve the position and movement of particles. Before exploring a range of applications, you need to distinguish the scalar and vector quantities related to movement.

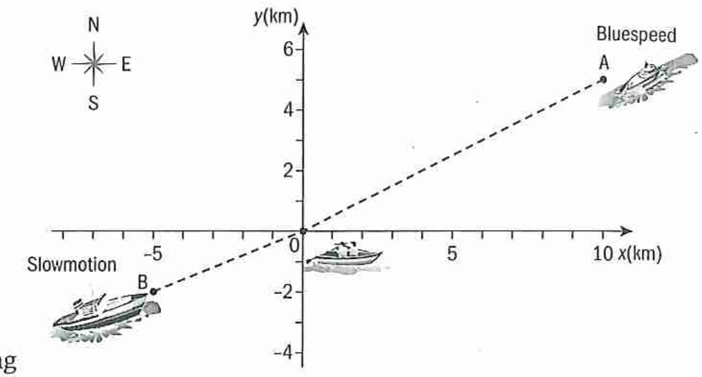
- **Distance** is a scalar quantity. **Displacement** is a vector quantity.
- **Speed** is a scalar quantity that considers only the magnitude. **Velocity** is a vector quantity that must consider both magnitude and direction.
- **Acceleration** is a 'change in velocity'. This change can be in the magnitude (speed) of the velocity or in the direction of the velocity.

Vector equations of lines provide a method for determining the position of an object when the parameter chosen is time, as shown in the next example.

Example 51

The diagram shows a boat in danger at the point O(0, 0) and the paths of two rescue boats, *Bluespeed* and *Slowmotion*, as they depart from the positions A(10, 5) and B(−5, −2) respectively.

Bluespeed moves at a speed of 15 km h^{-1} and *Slowmotion* moves at a speed of 8 km h^{-1}.



a Find an equation for the position of each boat t hours after departing from A and B respectively.

b Hence, determine how long it takes for each boat to reach the boat in danger.

Answers

a *Bluespeed* moves in the direction of $\overrightarrow{AO}$ at a speed of 15 km h^{-1}. So, its velocity vector is

$$\mathbf{v}_1 = \begin{pmatrix} -6\sqrt{5} \\ -3\sqrt{5} \end{pmatrix}$$ and a vector equation for its movement

is $$\mathbf{r} = \begin{pmatrix} 10 \\ 5 \end{pmatrix} + t\begin{pmatrix} -6\sqrt{5} \\ -3\sqrt{5} \end{pmatrix},\ t \geq 0.$$

Bluespeed:
Find the vector with magnitude 15 in the direction of $\overrightarrow{AO} = \begin{pmatrix} -10 \\ -5 \end{pmatrix}$ *and use* $\mathbf{r} = \mathbf{a} + t\mathbf{v}_1$, *where* $a = \overrightarrow{AO}$.

Slowmotion moves in the direction of $\overrightarrow{BO}$ at a speed of 8 km h^{-1}. So, its velocity vector is

$$\mathbf{v}_2 = \begin{pmatrix} \frac{40\sqrt{29}}{29} \\ \frac{16\sqrt{29}}{29} \end{pmatrix}$$ and a vector equation for its

movement is $$\mathbf{r} = \begin{pmatrix} -5 \\ -2 \end{pmatrix} + t\begin{pmatrix} \frac{40\sqrt{29}}{29} \\ \frac{16\sqrt{29}}{29} \end{pmatrix}\ t \geq 0$$

Slowmotion:
Find the vector with magnitude 8 in the direction of $\overrightarrow{BO} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ *and use* $\mathbf{r} = \mathbf{b} + t\mathbf{v}_2$, *where* $b = \overrightarrow{BO}$.
In both equations, parameter t represents the time, in hours, after departure to the rescue site.

b $$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix} + t\begin{pmatrix} -6\sqrt{5} \\ -3\sqrt{5} \end{pmatrix} \Rightarrow t = \frac{\sqrt{5}}{3}$$

*Using the vector equations obtained in part **a**, substitute the position of the boat in danger for* $\mathbf{r}$ *and solve for t.*

Bluespeed takes approximately 44 minutes and 43 seconds to reach the rescue site.

Convert the time in hours into minutes and seconds to obtain appropriate answers.

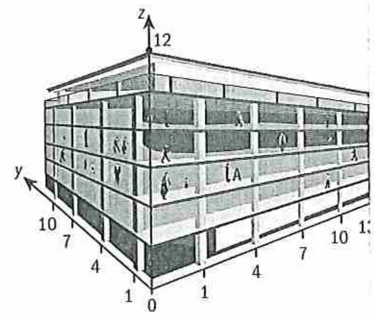
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \end{pmatrix} + t\begin{pmatrix} \frac{40\sqrt{29}}{29} \\ \frac{16\sqrt{29}}{29} \end{pmatrix} \Rightarrow t = \frac{\sqrt{29}}{8}$$

Slowmotion takes approximately 40 minutes and 23 seconds to reach the rescue site.

You can use your GDC to do this conversion.

Example 52

The diagram shows the model of a building with dimensions 19 m by 10 m by 12 m. All the floors are 2 m high.
Anne departs from a position A(4, 3, 4) and moves toward the elevator whose path has equation $x = 10$ and $y = 5$.
The elevator moves along the intersection of these two planes.



a How many seconds does it take Anne to reach the elevator if she walks at a speed of 1.5 metres per second?

Anne takes the elevator down to the ground floor and then walks in the direction of the vector $\begin{pmatrix}-1\\-1\\0\end{pmatrix}$ towards a door.

b Given that Anne walks at a speed of 1.6 metres per second, determine the coordinates of th location of the door and the time it takes her to get there. State any assumptions you have ma

Answers

a Anne's location: A(4, 3, 4)
Elevator's location: E(10, 5, 4)

To determine the coordinates of the location of t. elevator, use the equation of the path of the elev and the z-coordinate of point A (i.e. assume that movement takes place in the plane $z = 4$).

$AE = \sqrt{6^2+2^2+0^2} = 2\sqrt{10}$ metres

Use $AE = |\overrightarrow{AE}|$
$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$t = \frac{2\sqrt{10}}{1.5} = 4.22$ seconds (to 3 sf)

Use $t = \frac{distance}{speed}$

So, Anne takes 4.22 seconds to get to the elevator.

b In the plane $x \circ y$, an equation of Anne's movement is:

$\mathbf{r} = \begin{pmatrix}10\\5\end{pmatrix} + t\begin{pmatrix}-0.8\sqrt{2}\\-0.8\sqrt{2}\end{pmatrix}$

Assume that the ground floor lies in the plane z : and reduce the problem to a 2-D situation. Henc location of the elevator is (10, 5) and Anne's vel has a direction of $\begin{pmatrix}-1\\-1\end{pmatrix}$ *and magnitude 1.6.*

$0 = 5 - 0.8\sqrt{2}t$

$\Rightarrow t = \frac{5}{0.8\sqrt{2}} = 4.42$ (to 3 sf)

$\Rightarrow \mathbf{r} = \begin{pmatrix}5\\0\end{pmatrix}$

Anne reaches the door when either $x = 0$ or $y =$ By inspection, $x = 0$ gives a negative y-value. So $y = 0$ to obtain the value of t and substitute it ir equation to obtain the coordinates of the door.

So, the door is located at (5, 0, 0) and Anne takes 4.42 seconds to get there.
You assumed that the locations of objects and people were represented by points and the floors were represented by planes.

(ercise 11S

Boat A's position is given by the parametric equations $x = 3 - t$, $y = 2t - 4$ where position is in km and time in hours. Boat B's position is given by $x = 4 - 3t$, $y = 3 - 2t$.

a Find the initial position of each boat.

b Find the velocity vector of each boat.

c What is the angle between the paths of the boats?

d At what time are the boats closest to each other?

Why might it be argued that vector equations of lines are superior to Cartesian ones?

(AM-STYLE QUESTION

The position vector at time t of a particle P moving in 3-D space is given by

$\overrightarrow{OP} = (5+10t)\mathbf{i}+(20-20t)\mathbf{j}+(30t-10)\mathbf{k},\ t \geq 0.$

a Find the coordinates of P when $t = 0$.

b Show that P moves along the line L with Cartesian equations

$$\frac{x-5}{1}=\frac{y-20}{-2}=\frac{z+10}{3}.$$

c **i** Find the value of t when P lies on the plane with equation $x + y + z = 55$.

ii State the coordinates of P at this time.

iii Hence, find the total distance travelled by P before it meets the plane.

The position vector at time t of another particle, Q, is

given by $\overrightarrow{OQ} = \begin{pmatrix} 2t^2 \\ 1-2t \\ 1+t^2 \end{pmatrix}$ $t \geq 0$.

d **i** Find the value of t for which the distance between the particles P and Q is a minimum.

ii Find the coordinates of P and Q at this time.

e Let **a**, **b** and **c** be the position vectors of Q at times $t = 0$, $t = 1$ and $t = 2$ respectively.

i Show that **a** – **b** and **b** – **c** are non-collinear.

ii What is the geometric meaning of part **i**?

3 The diagram shows the unit cube OABCDEFG.
Let O be the origin, OA the x-axis, OC the y-axis and OD the z-axis.
Let P, Q and R be the midpoints of [BC], [CG] and [DG] respectively.

a Find the position vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{OR}$ in component form.

b Find the Cartesian equation of the plane PQR.
Let S, T and U be the midpoints of [ED], [AE] and [AB] respectively.

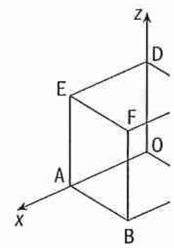
c Show that PQRSTU is a regular hexagon and find its area.

d Show that the line OF is perpendicular to the plane PQR.

e Determine the coordinates of the point I where the line OF meets the plane PQR.

f Hence, find the distance from F to the hexagon PQRSTU.

The edges of
cube have len



Explore further vectors

In this chapter, we have looked at two approaches to the study of vectors – geometric and analytic. An alternative is the **axiomatic** approach, where no attempt is made to describe the nature of vectors or the algebraic operations on them. Instead, vectors and operations are seen as undefined concepts of which we know nothing except that they satisfy a set of axioms. Such a system, with appropriate axioms, is called a **linear space**. The axiomatic point of view is mathematically the most satisfactory as it is independent of particular geometric representations, systems of coordinates and dimensions. Examples of linear spaces occur in many different branches of mathematics.

1 Examine linear spaces and the axioms that define them.
2 Explore the inner product and its relationship to the distance between points and the way angles are measured.
3 Explore alternative definitions of distance in the plane and the consequences on the geometry of shapes.

An **axiom** is
statement wh
assumed to b
without need
proof, used as
for developing
argument.

What is a dist
can we define
between two p
different ways

Review exercise

1 Given $\mathbf{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, find

a $3\mathbf{u} - 2\mathbf{v}$ **b** $|\mathbf{u}|$, $|\mathbf{v}|$, and $|\mathbf{u} + \mathbf{v}|$

2 Express $\mathbf{u} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ as a linear combination of $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$.

3 Let $\mathbf{a} = \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$

a Find the unit vector in the direction of $\mathbf{a}$.

b Find the vectors with magnitude 5 collinear with $\mathbf{a}$.

4 Consider the vector $\mathbf{u} = \cos\alpha\cos\beta\mathbf{i} + \sin\alpha\cos\beta\mathbf{j} + \sin\beta\mathbf{k}$
Show that $\mathbf{u}$ is a unit vector.

EXAM-STYLE QUESTIONS

5 Let $\mathbf{u} = \mathbf{i} + \tan\alpha\,\mathbf{j}$ and $\mathbf{v} = \tan\beta\,\mathbf{i} + \mathbf{j}$, where $0 < \alpha, \beta < \frac{\pi}{4}$.
Let γ be the angle between the vectors **u** and **v**.
Show that $\alpha + \beta + \gamma = \frac{\pi}{2}$.

6 Let θ be the angle between the unit vectors **a** and **b**, where $0 \le \theta \le \pi$.

a Express $|\mathbf{a} - \mathbf{b}|^2$ and $|\mathbf{a} + \mathbf{b}|^2$ in terms of θ.

b Hence, determine the exact value of $\sin\theta$ for which $|\mathbf{a} + \mathbf{b}| = 2|\mathbf{a} - \mathbf{b}|$

7 Write down, in vector and parametric form, equations of the plane passing through the point A and parallel to the vectors **p** and **q** when:

a A(2, 3, 4), $\mathbf{p} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{q} = \mathbf{j} - 2\mathbf{k}$

b A(1, 0, −2), $\mathbf{p} = -2\mathbf{i} + \mathbf{k}$ and $\mathbf{q} = -\mathbf{j}$

EXAM-STYLE QUESTION

8 The vector equations of the lines L_1 and L_2 are given by:

$$L_1 : \mathbf{r}_1 = \begin{pmatrix} 5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$L_2 : \mathbf{r}_2 = \begin{pmatrix} -2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

The lines intersect at the point P. Find the coordinates of P.

Review exercise

EXAM-STYLE QUESTIONS

1 Find the size of the angle between the vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$
Give your answer to the nearest degree.

2 Write down vector, parametric and Cartesian equations of the planes containing the points A, B and C.

a A(1, 1, 0), B(−1, 1, 2) and C(0, 1, −1)

b A(−1, 1, 1), B(−2, −1, 2) and C(0, 1, −1)

3 Write down, in scalar product form, the equation of the plane passing through the point with position vector **a** and with normal vector **n** when:

a $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{n} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

b $\mathbf{a} = \mathbf{i} + 2\mathbf{k}$ and $\mathbf{n} = \mathbf{i} - \mathbf{j}$

4 Find a Cartesian equation of the plane that:

a contains the point A(2, −3, 4) and is perpendicular to the position vector of A

b contains the points A(6, 0, 0), B(0, 0, −3) and C(3, 6, 0)

c contains the points A(3, 2, −1) and B(4, 4, 0) and is perpendicular to the plane $2x + 4y - 4z = 3$

d contains the points A(2, −1, −3) and B(4, −3, 2) and is parallel to the x-axis

e passes through the point (3, 4, 2) and is perpendicular to the x-axis

f passes through the point (3, 2, 1) and is perpendicular to each of the planes $2x + 3y - z = 5$ and $3x + 3z = 2$

g passes through the point (3, 1, 1) and contains the line of intersection of the planes $x + y + 5z = 0$ and $2x + 3y + 12z = 0$

EXAM-STYLE QUESTION

5 Calculate the acute angle between the lines with equations $\mathbf{r} = \mathbf{i} + \mathbf{j} + \alpha(2\mathbf{i} - \mathbf{j} - \mathbf{k})$ and $\mathbf{r} = \mathbf{k} + \beta(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.

6 The vector equations of the lines L_1 and L_2 are given by:

L_1: $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

L_2: $\mathbf{r} = \mathbf{i} + 4\mathbf{j} + 5\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

Show that the lines are concurrent.

Hence, determine a vector equation of the plane that contains these lines.

Concurrent li
are lines that al
through a certai
point.

7 Solve the system $\begin{cases} x + y + z = 3 \\ 2x + y - 2z = 0 \\ 3x - 2y + 5z = 23 \end{cases}$

and explain the geometric meaning of your answer.

EXAM-STYLE QUESTION

8 Consider the following system of equations

$$\begin{cases} 3x + y + z = 1, \\ x + y - z = 4 \\ 2x + y + bz = a \text{ where } a \text{ and } b \text{ are constants.} \end{cases}$$

a Solve the system in terms of a and b.

b Hence, write down the values of a and b for which this system of equations has a non-unique solution and state its geometric meaning.

CHAPTER 11 SUMMARY

Geometric vectors and basic operations

- A vector is defined by direction and magnitude.
- In mathematics, $\overrightarrow{AB}$ usually represents either the **position vector** of B relative to A or the **displacement** from A to B. A is called the **initial point** and B the **terminal point**.
- When we work with vectors you need to distinguish the vector from its magnitude, which is a non-negative scalar.

  The magnitude of $\overrightarrow{AB}$ is simply the length AB and is denoted by $|\overrightarrow{AB}|$. If $\overrightarrow{AB} = \mathbf{a}$ then you can represent its magnitude by $|\mathbf{a}|$ or simply by a.
- **Special case**

  The vector $\overrightarrow{AA}$ has no defined direction and is represented by a single point A. This vector is called a **null vector** or **zero vector** and it is the only vector that has a magnitude of zero.
- The sum of two vectors is determined by the parallelogram law: $\overrightarrow{BA} + \overrightarrow{BC} = \overrightarrow{BD}$ where ABCD is a parallelogram.
- This definition is called the **triangle law** and has the advantage that it can easily be applied to the addition of several vectors.
- **Properties of vector addition**

  Commutative property: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

  Associative property: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

  Additive identity property (the zero vector): $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

  Additive inverse property (the opposite vector):

  $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$
- The difference of two vectors **u** and **v** is the vector obtained when you add **u** to the opposite of **v**.
- The product of a vector **u** and a positive scalar k is another vector **v** with the same direction as **u** and magnitude $|\mathbf{v}| = k|\mathbf{u}|$.
- In general, the product of a vector **u** and a negative scalar k is another vector **v** with the opposite direction to **u** and magnitude

  $$|\mathbf{v}| = |k|\,|\mathbf{u}| = -k|\mathbf{u}|.$$
- **Properties of scalar multiplication**

  Commutative property: $\alpha\mathbf{v} = \mathbf{v}\,\alpha$

  Associative property: $\alpha(\beta\mathbf{v}) = (\alpha\beta)\,\mathbf{v}$

  Distributive property (1): $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$

  Distributive property (2): $(\alpha + \beta)\,\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$

  Multiplicative identity property: $1\mathbf{v} = \mathbf{v}$

  Property of zero: $0\mathbf{v} = \mathbf{0}$ and $\alpha\mathbf{0} = \mathbf{0}$

Continued on next page

Vector algebra

- In 2-D space, $\mathbf{i} = \begin{pmatrix}1\\0\end{pmatrix}$, $\mathbf{j} = \begin{pmatrix}0\\1\end{pmatrix}$ and O(0, 0).
- In 3-D space, $\mathbf{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\mathbf{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

  The origin O has coordinates (0, 0, 0)
- Given two vectors in the plane, $\mathbf{u} = \begin{pmatrix}u_1\\u_2\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}v_1\\v_2\end{pmatrix}$, and a real number λ:
  - The sum of the two vectors **u** and **v** is defined by

    $\mathbf{u} + \mathbf{v} = \begin{pmatrix}u_1\\u_2\end{pmatrix} + \begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}u_1+v_1\\u_2+v_2\end{pmatrix}$
  - The product of a scalar λ and a vector **u** is defined by $\lambda\mathbf{u} = \begin{pmatrix}\lambda u_1\\\lambda u_2\end{pmatrix}$
  - The zero vector or null vector is $\mathbf{0} = \begin{pmatrix}0\\0\end{pmatrix}$
  - The opposite vector of $\mathbf{u} = \begin{pmatrix}u_1\\u_2\end{pmatrix}$ is $-\mathbf{u} = \begin{pmatrix}-u_1\\-u_2\end{pmatrix}$
- If $\mathbf{v} = \begin{pmatrix}v_1\\v_2\end{pmatrix}$ then the magnitude of v is given by

  $v = |\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$
- If $\mathbf{v} = \begin{pmatrix}v_1\\v_2\end{pmatrix}$, the **unit vector** in the direction of a non-zero vector **v** is $\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \begin{pmatrix}\frac{v_1}{\sqrt{v_1^2+v_2^2}}\\ \frac{v_2}{\sqrt{v_1^2+v_2^2}}\end{pmatrix}$.
- Two vectors $\mathbf{u} = \begin{pmatrix}u_1\\u_2\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}v_1\\v_2\end{pmatrix}$ are **collinear** if $\mathbf{u} = k\mathbf{v}$ or $\mathbf{v} = k\mathbf{u}$ for some scalar k

  If $k > 0$, **u** and **v** have the same direction;

  if $k < 0$, **u** and **v** have opposite directions.

Continued on

:tors, points and equations of lines

ıree points A, B and C in the plane are collinear when $\overrightarrow{AB}$ ıd $\overrightarrow{AC}$ are collinear vectors, that is when $\overrightarrow{AC} = k\overrightarrow{AB}$ for ıme scalar k.

wo points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, have **displacement vector**

$$\overrightarrow{B} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ where } x = x_2 - x_1,\ y = y_2 - y_1 \text{ and } z = z_2 - z_1$$

'ou can also assign a **position vector** to each point

$$\overrightarrow{A} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \text{ and } \overrightarrow{OB} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

'he magnitude of a vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is given by $v = |\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

Given two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the distance between A and B is given by the magnitude of the vector $\overrightarrow{AB}$

$$AB = |\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The unit vector in the direction of a non-zero vector $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \begin{pmatrix} \frac{v_1}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \\ \frac{v_2}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \\ \frac{v_3}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \end{pmatrix}$$

Two vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ are collinear if $\mathbf{u} = k\mathbf{v}$ *or* $\mathbf{v} = k\mathbf{u}$ for some scalar k. If $k > 0$, $\mathbf{u}$ and $\mathbf{v}$ have the same direction; if $k < 0$, $\mathbf{u}$ and $\mathbf{v}$ have opposite directions.

In general, the coordinates of the midpoint M of a line segment [AB], with $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, are given by

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Continued on next page

- A point R is on the line AB when $\overrightarrow{AR} = \lambda\overrightarrow{AB}$ for some real value of λ. This is called a **vector equation of the line** AB. If the point R has position vector $\mathbf{r}$, the point A has position vector $\mathbf{a}$, and $\overrightarrow{AB} = \mathbf{u}$, the direction vector, then the vector equation $\overrightarrow{AR} = \lambda\overrightarrow{AB}$ can be re-written as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$
- If the line AB lies in the plane $x \cdot y$, then you can represent the vector in component form $\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ This gives a pair of **parametric equations**

  $x = x_1 + \lambda u_1$ and $y = y_1 + \lambda u_2$

  or $\lambda = \frac{x - x_1}{u_1} = \frac{y - y_1}{u_2}$

  If both components of the vector are non-zero, eliminate the parameter λ to obtain a **Cartesian equation** of the line (AB):

  $$y - y_1 = \frac{u_2}{u_1}(x - x_1)$$

  This can be reduced to the form $y = mx + c$ where $m = \frac{u_2}{u_1}$ is the gradient of the line and $c = y_1 - \frac{u_2}{u_1}x_1$ is the y-intercept.

- If the line AB lies in 3-D space, $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, the vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$ can be transformed into three parametric equations

  $x = x_1 + \lambda u_1$, $y = y_1 + \lambda u_2$ and $z = z_1 + \lambda u_3$

  where $\lambda = \frac{x - x_1}{u_1} = \frac{y - y_1}{u_2} = \frac{z - z_1}{u_3}$

  If all the components of the vector are non-zero, eliminate the parameter λ to obtain Cartesian equations of the line AB

  $$\frac{x - x_1}{u_1} = \frac{y - y_1}{u_2} = \frac{z - z_1}{u_3}$$

Continued on

Scalar product

- Given two non-zero vectors **u** and **v**, $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$, where θ is the angle between **u** and **v**.
- Here are some important consequences of the geometric definition of the scalar product.
  - The scalar product of two vectors is always a number.
  - The definition of scalar product does not depend on the dimensions of the space.
  - $\mathbf{u}\cdot\mathbf{v} = 0$ if and only if $\mathbf{u} = \mathbf{0}$, $\mathbf{v} = \mathbf{0}$ or **u** and **v** are **orthogonal**.
  - $\mathbf{u}\cdot\mathbf{v} = \pm|\mathbf{u}|\,|\mathbf{v}|$ if **u** and **v** are parallel.
  - $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$
  - $\mathbf{u}\cdot\mathbf{v} > 0$ when q is acute and $\mathbf{u}\cdot\mathbf{v} < 0$ when θ is obtuse.
  - $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
  - $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$
  - $(\lambda\mathbf{u})\cdot\mathbf{v} = \lambda(\mathbf{u}\cdot\mathbf{v})$
- Given two vectors in the plane, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$,

  $$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2$$

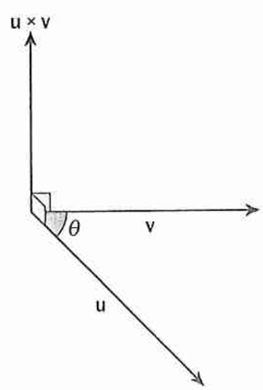
  In 3-D space, given two vectors, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$,

  $$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$
- $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$, where θ is the angle between the vectors **u** and **v**.

Vector product

- Given $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, the **vector (cross) product** of **u** and **v** is the vector given by $\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$.
- $\mathbf{u}\times\mathbf{v} = (|\mathbf{u}|\,|\mathbf{v}|\sin\theta)\,\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit vector orthogonal (normal) to both **u** and **v** whose direction is given by the right-hand rule illustrated in the diagram.
- Given a parallelogram ABCD, if $\mathbf{u} = \overrightarrow{AB} = \overrightarrow{DC}$ and $\mathbf{v} = \overrightarrow{AD} = \overrightarrow{BC}$, the area of ABCD is numerically equal to $|\mathbf{u}\times\mathbf{v}|$

  The area of triangle ABD equals $\frac{1}{2}|\mathbf{u}\times\mathbf{v}|$
- Summary of the algebraic properties of the vector (cross) product
  - $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$
  - $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}$
  - $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = \mathbf{0}$ ($\mathbf{u}\times\mathbf{v}$ is orthogonal to **u**)



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- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \mathbf{0}$ ($\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{v}$)
- $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are collinear.
- $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
- $(\lambda\mathbf{u}) \times \mathbf{v} = \lambda(\mathbf{u} \times \mathbf{v})$
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Vectors and equations of planes

- In this case, any point on the plane with position vector $\mathbf{r}$ satisfies the **vector equation of a plane**
  $\mathbf{r} = \mathbf{a} + \alpha\mathbf{u} + \beta\mathbf{v}$
- If the parameters α and β are eliminated, an equation of the form $ax + by + cz = d$ is obtained.
  This equation is called a **Cartesian equation of the plane.**
- $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$
  or $ax + by + cz = d$ where $d = ax_1 + by_1 + cz_1$
  Equation of the plane (using the normal vector)
  $\begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$ this can also be written as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}$
  or $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$
- In the plane, given two lines with gradients m_1 and m_2, we can calculate the angle θ between them using the formula $\theta = |\beta - \alpha|$ where $\alpha = \arctan m_1$ and $\beta = \arctan m_2$
- The angle θ between a line parallel to $\mathbf{u}$ and a plane with normal vector $\mathbf{n}$ is given by
  $\theta = \arcsin\left(\left|\frac{\mathbf{u} \cdot \mathbf{n}}{|\mathbf{u}||\mathbf{n}|}\right|\right), 0 \le \theta \le 90°$
- The angle θ between a plane with normal vector $\mathbf{m}$ and a plane with normal vector $\mathbf{n}$ is equal to the angle between the lines in the direction of the vectors $\mathbf{m}$ and $\mathbf{n}$. It is given by
  $\theta = \arccos\left(\left|\frac{\mathbf{m} \cdot \mathbf{n}}{|\mathbf{m}||\mathbf{n}|}\right|\right), 0 \le \theta \le 90°$
- The **distance from a point to a plane** is measured along the perpendicular to the plane that contains the point.