b $P(T \cap C') = \frac{25}{53}$

Probability that staff member drinks tea but not coffee is $\frac{25}{53}$

36 − 11 = 25

c $P(C \mid T) = \frac{P(C \cap T)}{P(T)} = \frac{\frac{11}{53}}{\frac{36}{53}}$

$= \frac{11}{\cancel{53}} \times \frac{\cancel{53}}{56} = \frac{11}{36}$

Probability that staff member is a tea drinker who drinks coffee is $\frac{11}{36}$

d $P(C' \mid T) = \frac{P(C' \cap T)}{P(T)} = \frac{\frac{25}{53}}{\frac{36}{53}}$

$= \frac{25}{\cancel{53}} \times \frac{\cancel{53}}{36} = \frac{25}{36}$

$P(C' \cap T) = P(T \cap C')$

Exercise 6J

EXAM-STYLE QUESTIONS

1 There are 27 students in a class. 15 take Art and 20 take Theater. Four do neither subject. How many students do both subjects? One person is chosen at random. Find the probability that

a he or she takes Theater but not Art

b he or she takes at least one of the two subjects

c he or she takes Theater, given that they take Art.

2 For events A and B it is known that: $P(A' \cap B') = 0.35$, $P(A) = 0.25$ and $P(B) = 0.6$ Find

a $P(A \cap B)$ **b** $P(A \mid B)$ **c** $P(B' \mid A')$

3 48% of all teenagers own a skateboard and 39% of all teenagers own a skateboard and roller blades. What is the probability that a teenager owns roller blades given that the teenager owns a skateboard?

4 A number is chosen at random from this list of eight numbers:

1 2 4 7 11 16 22 29

Find:

a P(it is even | it is not a multiple of 4)

b P(it is less than 15 | it is greater than 5)

c P(it is less than 5 | it is less than 15)

d P(it lies between 10 and 20 | it lies between 5 and 25)

5 In my town 95% of all households have a desktop computer. 61% of all households have a desktop computer and a laptop computer. What is the probability that a household has a laptop computer given that it has a desktop computer?

6 The probability that a student takes Technology and Spanish is 0.1. The probability that a student takes Technology is 0.6. What is the probability that a student takes Spanish given that the student is taking Technology?

7 U and V are mutually exclusive events. $P(U) = 0.26$ and $P(V) = 0.37$. Find:

a $P(U \text{ and } V)$

b $P(U \mid V)$

c $P(U \text{ or } V)$

8 A teacher gave her class an IB Paper 1 and an IB Paper 2. 35% of the class passed both tests and 52% of the class passed the first test. What percent of those who passed the first test also passed the second test?

9 A jar contains black and white marbles. Two marbles are chosen without replacement. The probability of selecting a black marble and then a white marble is 0.34, and the probability of selecting a black marble on the first draw is 0.47. What is the probability of selecting a white marble on the second draw, given that the first marble drawn was black?

EXAM-STYLE QUESTION

10 The table shows the number of left- and right-handed table-tennis players in a sample of 50 males and females.

	Left-handed	Right-handed	Total
Male	5	32	37
Female	2	11	13
Total	7	43	50

A table-tennis player was selected at random from the group. Find the probability that the player is:

a male and left-handed

b right-handed

c right-handed, given that the player selected is female.

11 Your neighbor has two children. You learn that he has a son, Sam. What is the probability that Sam's sibling is a brother?

This is not as obvious as it might seem!

Example 20

Maika goes on a photo safari. The probability that Maika takes a photo of a cheetah on any day is 0.3. Find how many days the safari must last so that the probability that Maika takes at least one photo of a cheetah will exceed 95%.

Answer

If the safari is for n days

$1-0.7^n \geq 0.95$

$0.05 \geq 0.7^n$

Use the complementary event, 'Maika did not take a photo of cheetah in n days'.

$n\log(0.7) \leq \log(0.05)$

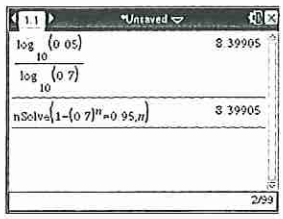
Take logarithms on both sides. Both logs are negative.

$n \geq \dfrac{\log(0.05)}{\log(0.7)} = 8.40$

The inequality can be solved immediately by a GDC.

The safari should last at least 9 days.

GDC help on CD:
Alternative demonstrations for the TI-84 Plus and Casio fx-9860GII GDCs are on the CD.



Exercise 6K

1 If $P(A) = 0.4$, $P(B) = 0.6$, $P(C) = 0.3$,
$P(A \cap B) = 0.24$, $P(B \cap C) = 0.15$ and $P(A \cup C) = 0.82$,
which of the events A, B and C are independent?
Give reasons for your answers.

2 A card is drawn from a standard deck of 52 cards.
Of these events:
- A 'the card is a Queen',
- B 'the color of the card is red'
- C 'the card is a face card'

which are independent? Explain your answer.

3 Given that A and B are independent events, show that:
a A and B' are independent
b A' and B are independent
c A' and B' are independent.

4 Given that $P(A) = \frac{1}{3}$, $P(A \cup B) = \frac{5}{6}$ and $P(B|A) = \frac{3}{4}$, find $P(B)$.
Are the events A and B independent?

5 Independent events A and B are such that $P(A) = 0.45$ and $P(A \cap B) = 0.18$. Find these probabilities:
a $P(B)$ **b** $P(A \cup B)$ **c** $P(A' \cap B')$

6 Independent events A and B are such that
$P(A) = a$, $P(B) = 2a$ and $P(A \cup B) = \frac{5}{8}$.
Find $P(A)$ and $P(B)$.

19 f c

7 In a game the players flip a coin and roll a dice. Draw a tree diagram to represent the game, and find the probabilities of these events:
- **a** 'a tail on the coin and a 6 on the dice'
- **b** 'a head on the coin and an even number on the dice'
- **c** 'no tail on the coin and a multiple of 3 on the dice.'

8 A combination lock on a suitcase consists of four digits. Notice that the leading digits can be zeros. Norbu is setting up the combination. What is the probability that his combination will be divisible by:
a 2 **b** 5 **c** 4?

9 A set contains a large number of integers. How many integers must you select so that the probability of selecting at least one odd integer is at least 0.92?

10 Julia plays tennis. She can score a point from the base line with a probability 0.55. How many times does she need to hit the ball so that the probability of Julia scoring a point exceeds 0.999?

6.9 Probability tree diagrams

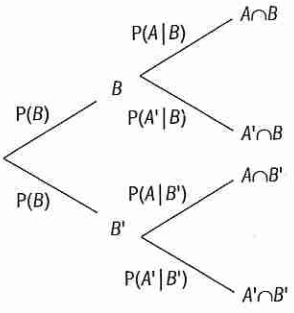
"Solving problems is a practical art, like swimming, or skiing, or playing the piano . . . if you wish to learn swimming you have to go in the water, and if you wish to become a problem solver you have to solve problems." – George Polya, *Mathematical Discovery* (1981).

George Polya (1887–1985) was born and educated in Hungary. He obtained a PhD in mathematics from Budapest and taught in Switzerland and at Brown, Smith and Stanford Universities in the United States. He was granted numerous honors and awards in mathematics, and taught and lectured in virtually every country of the world. He encouraged the use of problem solving techniques in learning mathematics and is best known for his outstanding book, the classic *How to Solve It* (1945).

Research Polya's steps to problem solving, which he outlines in *How To Solve It*.

Tree diagrams are useful tools for tackling probability problems involving two or more events. The next set of examples will show you how to use them.

Example 21 is a problem 'with replacement', i.e. the conditions for the second event are exactly the same as they were for the first event.



In the next example some branches of the tree diagram are shorter than others. Subsequent events depend on the outcomes of previous ones.

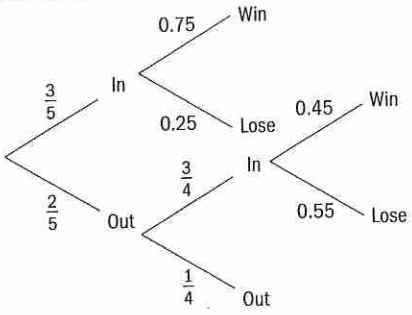
Example 24

Toby is a rising star of the school tennis club. He has found that when he gets his first serve in the probability that he wins that point is 0.75. When he uses his second serve there is a 0.45 chance of him winning the point. He is successful at getting his first serve in on 3 out of 5 occasions and his second serve in on 3 out of 4 occasions.

a Find the probability that the next time it is Toby's turn to serve he wins the point.

b Given that Toby wins the point, what is the probability that he got his first serve in?

Answers



On this tree diagram, it is not necessary to continue the branches once the point has been won.

a P (win) = (get first serve in and win) + (miss first serve, get second serve in and win)

$$\left(\frac{3}{5} \times 0.75\right) + \left(\frac{2}{5} \times \frac{3}{4} \times 0.45\right)$$

$$= 0.45 + 0.135$$

$$= 0.585$$

Multiply along the branches.

b P(first serve in | win)

P(first serve in and win point | win point)

$$= \frac{\left(\frac{3}{5} \times 0.75\right)}{0.585} = 0.769 \text{ (3 sf)}$$

Both of these values have been found in part **a**.

Exercise 6L

1 The probability that it will rain on a certain day is 0.2. The probability that Shikma arrives late for work if it's raining is 0.4, while the probability that she will be late for work when it's not raining is 0.1. Find the probability that on a given day it will rain and Shikma will not be late for work.

2 Shankari performs blood tests on a virus. The probability that the test gives a positive result is 0.85. The probability that the person does have the virus given that the test was positive is 0.98, and the probability that the person doesn't have the virus given that the test was negative is 0.12. What is the probability that Shankari's diagnosis is correct?

3 Jan shoots two free throws in a basketball game. The probability that he scores the first shot is 0.75. The probability that he misses the second shot given that he scored the first shot is 0.15. The probability that he is going to score the second shot given that he missed the first shot is 0.8. What is the probability that Jan scores only one shot?

4 Given that $P(A)=\frac{1}{3}, P(B|A)=\frac{3}{5}$ and $P(B|A')=\frac{1}{2}$ find

a $P(B')$

b $P(A' \cup B')$

5 There are 12 purple and 18 orange chips in a bag. We draw three chips from the bag without replacing them. Find the probability that:

a all the chips are orange

b there is at least one purple chip

c there are more orange chips.

6 Sam draws three cards from a standard deck of 52 cards without replacing them. Find the probability that all three cards are:

a red

b hearts

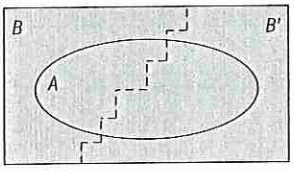
c of the same suit

d faces cards in the same suit.

Face cards are King, Queen, Jack.

6.10 Bayes' theorem

As you will see, Bayes' theorem enables you to solve more difficult probability problems where you need to analyze different possibilities. To visualize the situation you can use a Venn diagram.



The Venn diagram shows that event A may occur when event B occurs or A may occur when B does not occur. These are mutually exclusive situations. You can use this fact to write $P(A)$ as the sum of two expressions which correspond to the two situations.

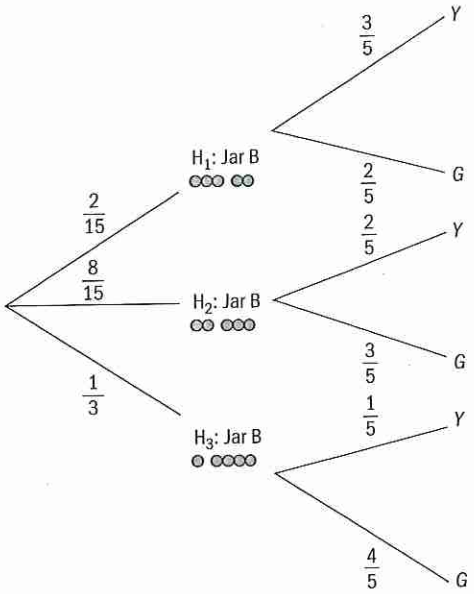
$$P(A)=P((A\cap B)\cup(A\cap B'))=P(B)\times P(A|B)+P(B')\times P(A|B')$$

This formula is called the **total probability formula**.

Example 28

There are two jars with purple and green marbles. In jar A there are 4 green and 6 purple marbles. Jar B contains 1 green and 2 purple marbles. We transfer 2 marbles from jar A to jar B and then draw a marble from jar B. A randomly drawn marble from the jar B is green. What is the probability that 2 marbles of different colors were transferred from jar A?

Answer



Draw the probability tree diagram

$$P(H_2|G) = \frac{P(H_2 \cap G)}{P(G)}$$

Apply the conditional probability formula.

$$= \frac{P(H_2) \times P(G|H_2)}{P(H_1) \times P(G|H_1) + P(H_2) \times P(G|H_2) + P(H_3) \times P(G|H_3)}$$

Apply the total probability formula.

$$= \frac{\frac{8}{15} \times \frac{2}{5}}{\frac{2}{15} \times \frac{3}{5} + \frac{8}{15} \times \frac{2}{5} + \frac{1}{3} \times \frac{1}{5}} = \frac{\frac{16}{75}}{\frac{27}{75}} = \frac{16}{27}$$

Exercise 6M

EXAM-STYLE QUESTION

1 Two boxes contain cards. In the first box there are 9 cards with numbers 1 to 9 written on them, and in the second box there are 5 cards with numbers 4 to 8 written on them. A box is randomly selected and a card is drawn from it.

a What is the probability that the card will show an even number?

b What is the probability that an even card was drawn from the first box?

EXAM-STYLE QUESTIONS

2 In a factory two machines produce bolts and the bolts are stored in a warehouse. The first machine produces 60% of all the bolts but 5% of them are defective while 2% of the bolts produced by the second machine are defective. A bolt is randomly selected from the warehouse.

a What is the probability that the bolt will be defective?

b What is the probability that the bolt was produced on the first machine given that it was defective?

3 40% of the children attending a summer camp are girls and 60% are boys. The children vote for their favorite sport. 75% of the girls and 35% of the boys vote for beach volleyball as their favorite sport. A child is randomly selected.

a What is the probability the child voted for beach volleyball as their favorite sport?

b What is the probability that beach volleyball was not their favorite sport given that the child was a girl?

4 In one box there are 14 white and 16 black balls, while in a second box there are 7 white and 12 black balls. A ball is drawn from the first box and placed in the second box, and then two balls are drawn from the second box.

a What is the probability that both balls are black?

b What is the probability that the ball drawn from the first box was white, given that both balls drawn from the second box were white?

5 A sample space U contains the events A and B. These probabilities are given: $\mathrm{P}(B)=\frac{2}{3}, \mathrm{P}(A|B)=\frac{5}{6}$ and $\mathrm{P}(A'|B')=\frac{1}{4}$

a Draw a probability tree diagram representing this information

b Find: **i** $\mathrm{P}(A)$ **ii** $\mathrm{P}(B|A)$ **iii** $\mathrm{P}(B'|A')$.

6 A pair of dice is in a non-transparent bag. One dice is biased and the probability of obtaining a 6 on that dice is $\frac{2}{3}$, while the other dice is unbiased. A dice is taken from the bag at random and roll it.

a What is the probability of obtaining a 6?

b What is the probability that the unbiased dice was taken given that you did not obtain a 6?

EXAM-STYLE QUESTIONS

7 A survey of a large group of adults shows that 18% have lung problems. Of these it is found that 70% are heavy smokers, 20% smoke occasionally and 10% are non-smokers. Of those who don't have lung problems it is found that 5% are heavy smokers, 15% smoke occasionally and 80% are non-smokers. An adult is selected at random from the group.

a Find the probability that the selected person is a non-smoker.

b Find the probability that the selected person has lung problems given that they are a heavy smoker.

8 There are three urns labelled A, B and C. In urn A there are 2 white and 4 red cubes, in urn B there are 5 white and 3 red cubes and urn C contains 4 white and 6 red cubes. An urn is selected at random and a cube is drawn from it.

a Find the probability that a red cube is drawn.

b Given that a red cube is drawn, find the probability that it is drawn from urn C.

9 On her way to school in the morning Anne can take three different routes, A, B and C. The probabilities that she takes routes A, B and C are 0.45, 0.20 and 0.35 respectively.
The probability that she will get to school on time if she takes route A is 0.95. If she takes route B it is 0.90 and the probability if she takes route C is 0.80.

a Find the probability that Anne will get to school on time.

b Find the probability that Anne took route A, given that she got to school on time.

c Find the probability that Anne took route B, given that she did not get to school on time.

10 There are two jars containing pink and brown marbles. In the first jar there are 5 pink and 10 brown marbles while in the second jar there are 4 pink and 5 brown marbles. We transfer 2 marbles from the first jar to the second jar and then draw a marble from the second jar.

a What is the probability that the marble drawn from the second jar is brown?

b What is the probability that we transferred 2 pink marbles from the first jar, given that a pink marble is drawn from the second jar?

c What is the probability that we transferred 2 brown marbles from the first jar, given that a pink marble is drawn from the second jar?

EXAM-STYLE QUESTIONS

11 In a company employees are organized in the following sections: management, production and marketing. 10% of all the employees are in management and 25% are in marketing. The percentages of female employees in management, production and marketing are 40%, 30% and 70% respectively.

a Find the probability that a randomly selected employee from this company is male.

b Given that a randomly selected employee is male find the probability that he works in the management section.

c Given that a randomly selected employee is female find the probability that she works in the marketing section.

12 Three machines produce nuts. The first machine produces 50% of all the nuts, the second machine produces 35% and the third machine produces 15%. Of the nuts produced by the first machine 4% are defective, versus 3% from the second machine and 6% from the third machine. Given that a randomly selected nut is not defective, find the probability that the nut was produced by the second machine.

13 There are 20 laptops in a classroom. 12 have a hard disk with a capacity of 160 GB and 8 with a capacity of 320 GB. A teacher randomly takes two laptops away to be tested. A student then takes a laptop for her project. What is the probability that the teacher takes both laptops with 320 GB given that the student takes one with 160 GB?

14 In a football team there are 11 players who can perform a penalty kick. 4 players are excellent and they score with a probability of 0.9, 4 have medium ability and they score with a probability of 0.6, whilst the remaining 3 are poor and they score with a probability of 0.2. A randomly selected player shoots the penalty kick. What is the probability that the player will score?

15 Two letters are drawn from the set $\{a, b, c, d, e, f, o\}$ without replacement. Then another letter is drawn. What is the probability that the third letter drawn is a vowel?

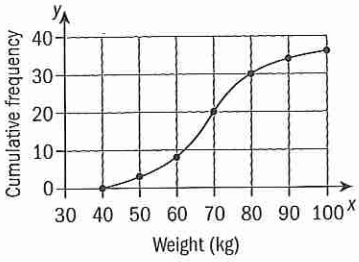
Review exercise

EXAM-STYLE QUESTIONS

1 The mode, median and mean of a set of positive integers are 6, 7 and 8 respectively. Find the smallest possible set of positive integers with the smallest variance that satisfies these conditions.

2 A and B are two independent events. Given that $P(B|A)=\frac{1}{3}$ and $P(A\cup B)=\frac{11}{12}$ find $P(A)$.

3 The weight, in kilograms, of students in a class is measured and the cumulative frequency diagram is shown.



- **a** Estimate the median weight of the students.
- **b** Estimate the middle 50% of the weight of the students.
- **c** How many students are in the class?
- **d** Construct the frequency distribution table and find the modal weight of the students in the group.

4 There are 7 boys and 5 girls in an environmental activity group. A committee of three members is selected from the group.

- **a** How many different committees can be selected?
- **b** Find the probability that Agatha and Jacob (two students from the group) are not both on the committee.
- **c** What is the probability that there will be more girls than boys on the committee?

5 Bassel invests in different companies. The probability that he invests in company X is $\frac{1}{3}$ and in company Y is $\frac{5}{9}$. The probability that he doesn't invest is $\frac{1}{9}$. The probability that the investments in companies X and Y yield a dividend are $\frac{3}{7}$ and $\frac{3}{5}$ respectively. The performances of the companies are independent.

- **a** Draw a probability tree diagram with the corresponding probabilities of all possible outcomes.
- **b** What is the probability that Bassel receives a dividend?
- **c** Find the probability that the dividend Bassel receives is from company Y.

6 There are 3 yellow, 4 blue and 5 green marbles in a bag. We randomly draw a marble from the bag and, without replacement, draw another marble. Given that the second marble drawn is green, what is the probability that the first marble was green too?

EXAM-STYLE QUESTIONS

7 A pair of unbiased dice is rolled and the product of the numbers showing on the upper faces is noted. Find the probabilities that the product is:

a A prime number

b An even number

c A number divisible by 3

d A number divisible by 6 given that it is an even number.

8 A class of 30 students takes a test on statistics and their results are such that: $\sum_{i=1}^{30} m_i = 540$ and $\sum_{i=1}^{30} m_i^2 = 9990$ where m_i represents the mark of an individual student. Calculate:

a **i** the mean mark in the test

ii the standard deviation of the marks in the test;

b Given that the pass mark for the test is 8, do you expect a student to fail the test?

9 Each odd number from 1 to $3n$, where n is odd, is written on a card and the cards are placed in a bag.

a How many cards are placed in the bag?

b What is the probability, in terms of n, that a card drawn randomly from the bag has a number divisible by 3?

Review exercise

1 There are four groups at a science conference. The heights of participants in each group are measured and the following mean heights of the groups are obtained.

Subject	Number of participants	Mean height(cm)
Biology	23	168
Chemistry	17	171
Environmental systems	8	163
Physics	20	177

Find the mean height of all the participants at the science conference.

2 **a** From the word STATISTICS how many arrangements of all the letters can be formed?

b What is the probability that the arrangement will start with the letter S?

c What is the probability that the arrangement will not end with a vowel?

3 A code lock consists of three digits. What is the probability that the randomly chosen code is:

a an even three-digit number

b a number divisible by 7

c a perfect square?

4 The probability that it rains on a particular day during July on the island of Hvar is 0.07. The hotel owner will give a 50% discount for a room on a rainy day in the month of July. David rents a room in the hotel for two days. Given that the rate for a room in the hotel is €85 per day, find the probability that for those two days David will pay less than €170.

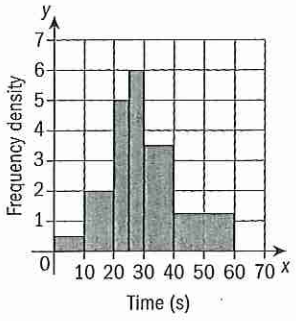
5 Electric car batteries are tested and the distances obtained on a single charge are given in the table.

Distance (km)	Frequency
$0 \leq d < 100$	2
$100 \leq d < 200$	5
$200 \leq d < 300$	7
$300 \leq d < 400$	12
$400 \leq d < 500$	10
$500 \leq d < 600$	4

a Find an estimate of the mean distance traveled on the batteries.

b Find estimate of the standard deviation of the distance traveled on the batteries. Estimate the maximum distance the manufacturer can claim with 95% certainty that the car would travel on a single charge.

6 A group of 140 competitors were each given a puzzle to complete. The times taken to do this were recorded. The diagram is the histogram of the scores.



a Construct a frequency distribution table.

b Two competitors from the group were selected at random. Find the probability that:

i both managed to assemble the puzzle within 20 seconds

ii no competitor was able to do it in less than 10 seconds.

c Find the estimations of the mean and standard deviation of times taken to assemble the puzzle.

7 How many times would you need to roll a pair of dice so that the probability of obtaining a sum of 7 at least once is greater than 0.95?

8 The red blood cell (RBC) count is a blood test which determines the number of red blood cells per micro-litre in a blood sample. The normal range for children is typically between 3.8 and 5.5 million per micro-litre. The results of the blood test for 120 children are given in the table.

RBC	Frequency
$3.4 < n \leq 3.8$	7
$3.8 < n \leq 4.2$	15
$4.2 < n \leq 4.6$	36
$4.6 < n \leq 5.0$	22
$5.0 < n \leq 5.4$	27
$5.4 < n \leq 5.8$	13

a Find the mean and the standard deviation of the RBC count.

b Construct the cumulative frequency distribution and draw the cumulative frequency diagram to estimate the median result of the RBC count.

c It is known that a higher RBC count is found in people who live at high altitudes. Given that all the children are healthy use your graph in **b** to estimate how many tested children live at high altitudes.

9 A school buys 20 new books. There are 6 books on statistics, 7 books on calculus, 4 books on geometry and 3 books on discrete mathematics. Given that the books are randomly put on the shelf in a classroom, find the probability that

a all the books on statistics will be at the beginning of the shelf

b all the books on calculus will be together.

10 The local basketball league consists of 12 teams. Team A is ranked fifth and the probability it will win against any of the higher ranked teams is 0.4. The probability it will win against the teams ranked in sixth to eighth place is 0.55 and of winning against the teams ranked ninth to twelfth is 0.75. Keith doesn't know which team team A will play against in the next game, but he would like to bet on A winning.

a What is the probability that Keith will win the bet?

b Given that Keith has lost the bet, what is the probability that team A played against a team with a better rank?