Section 6.4 — Vector Proofs in Geometry

We need to be able to solve a problem in several different ways. Sometimes one solution may be more direct and easier than another. If we try different methods of solution, we gain insight into the mathematical principles involved and increase our confidence in the results.

Euclidean proofs are the usual way to establish the properties of geometrical figures. The use of vectors is an alternate way to accomplish the same result.

There are two distinct approaches that can be taken when we use vectors to do proofs. One approach is to use point-to-point vectors. The other is to use position vectors. Point-to-point vectors usually lie in the plane of a figure and join one point of the figure to another. **Position vectors**, on the other hand, point from some outside origin, which is not usually part of the figure, to points in the figure.

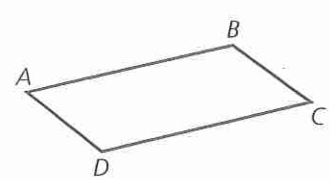
The two methods are illustrated in Examples 1 and 2 below. Remember that there are several things to do before you can actually start a proof. For instance, the proposition to be proved is usually expressed in words, so your first job is to express what is given and what is to be proved in the form of vector formulas or equations. To do this, you will need a suitably labelled diagram.

EXAMPLE 1

Two of the opposite sides of a quadrilateral are parallel and equal in length. Using point-to-point vectors, prove that the other two opposite sides are also parallel and equal in length.

Solution

Let $ABCD$ be a quadrilateral in which $AB = CD$ and $AB \parallel CD$. Using vectors, we write $\overrightarrow{AB} = \overrightarrow{DC}$. Likewise, what is to be proved can be written $\overrightarrow{AD} = \overrightarrow{BC}$.



Then
$$\begin{aligned}\overrightarrow{AD} &= \overrightarrow{AB} + \overrightarrow{BD}\\ &= \overrightarrow{DC} + \overrightarrow{BD}\\ &= \overrightarrow{BD} + \overrightarrow{DC}\\ &= \overrightarrow{BC}\end{aligned}$$

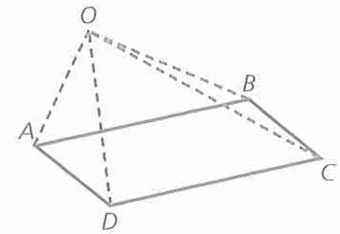
Therefore, if two of the opposite sides of a quadrilateral are parallel and equal, so are the other two opposite sides.

EXAMPLE 2

Two of the opposite sides of a quadrilateral are parallel and equal in length. Using position vectors, prove that the other two opposite sides are also parallel and equal in length.

Solution

Let $ABCD$ be a quadrilateral having $AB = CD$ and $AB \parallel CD$. Let O be an origin that is not in the plane of the quadrilateral.



As in Example 1, $\overrightarrow{AB} = \overrightarrow{DC}$ is given, and $\overrightarrow{AD} = \overrightarrow{BC}$ is to be proved.

Since
$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{DC} \\ \overrightarrow{OB} - \overrightarrow{OA} &= \overrightarrow{OC} - \overrightarrow{OD} \\ \overrightarrow{OD} - \overrightarrow{OA} &= \overrightarrow{OC} - \overrightarrow{OB} \\ \overrightarrow{AD} &= \overrightarrow{BC} \end{aligned}$$

The conclusion is the same as that of Example 1.

Sometimes a proof using position vectors requires the use of the division-point formula and the concept of linear independence. An example of that kind of proof is shown next.

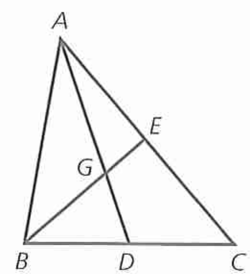
EXAMPLE 3

Prove that the medians of a triangle intersect at a point that divides each median in the ratio 2:1.

Solution

In ΔABC, D and E are the midpoints of BC and AC, respectively. If O is a point not in the plane of the triangle, then

$$\overrightarrow{OD} = \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OC}, \quad \text{and} \quad \overrightarrow{OE} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OC}$$



Let
$$\overrightarrow{OG} = k\overrightarrow{OA} + l\overrightarrow{OD}, \ (k + l = 1)$$
Then
$$= k\overrightarrow{OA} + l(\tfrac{1}{2}\overrightarrow{OB} + \tfrac{1}{2}\overrightarrow{OC})$$
$$= k\overrightarrow{OA} + \frac{1}{2}\,l\overrightarrow{OB} + \frac{1}{2}\,l\overrightarrow{OC}$$
Similarly
$$\overrightarrow{OG} = m\overrightarrow{OB} + n\overrightarrow{OE}, \ (m + n = 1)$$
$$= m\overrightarrow{OB} + n(\tfrac{1}{2}\overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{OC})$$
$$= \frac{n}{2}\overrightarrow{OA} + m\overrightarrow{OB} + \frac{n}{2}\overrightarrow{OC}$$

These two expressions for $\overrightarrow{OG}$ must be equal. Therefore,

$$k\overrightarrow{OA} + \frac{l}{2}\overrightarrow{OB} + \frac{l}{2}\overrightarrow{OC} = \frac{n}{2}\overrightarrow{OA} + m\overrightarrow{OB} + \frac{n}{2}\overrightarrow{OC}$$
or
$$(k - \tfrac{n}{2})\overrightarrow{OA} + (\tfrac{l}{2} - m)\overrightarrow{OB} + (\tfrac{l}{2} - \tfrac{n}{2})\overrightarrow{OC} = \vec{0}$$

Since the vertices of the triangle A, B, and C are not collinear, the position vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are not coplanar. Therefore, they are linearly independent vectors. This linear combination can only equal $\vec{0}$ if each of the coefficients separately equals zero:

$$k - \frac{n}{2} = 0, \frac{l}{2} - m = 0, \frac{l}{2} - \frac{n}{2} = 0$$

Since $k - \frac{n}{2} = 0, n = 2k$

Since $\frac{l}{2} - \frac{n}{2} = 0, l = n = 2k$

Now $k + l = 1$, so $k + 2k = 1$, or $k = \frac{1}{3}$

Then $l = \frac{2}{3}$

Then $\overrightarrow{OG} = \frac{1}{3}\overrightarrow{OA} + \frac{2}{3}\overrightarrow{OD}$, and G divides AD in the ratio 2:1.

Similarly, $\frac{l}{2} - m = 0, l = 2m$

and $\frac{l}{2} - \frac{n}{2} = 0, l = n = 2m$

Since $m + n = 1$, then $m = \frac{1}{3}, n = \frac{2}{3}$

Then $\overrightarrow{OG} = \frac{1}{3}\overrightarrow{OB} + \frac{2}{3}\overrightarrow{OE}$, and G divides BE in the ratio 2:1.

If you repeat this work using $\overrightarrow{AD}$, for instance, and the third median $\overrightarrow{CF}$, the result is the same. So the point of intersection G divides each of the medians in the ratio 2:1. G is called the **centroid** of the triangle.

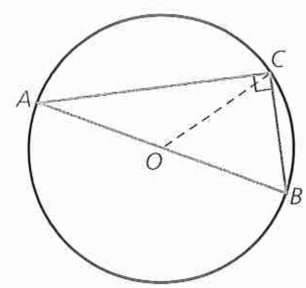
Another type of problem asks for a proof that two line segments are perpendicular. Such problems are handled by showing that the dot product of the corresponding vectors is zero.

EXAMPLE 4

Prove that an angle inscribed in a semicircle is a right angle.

Solution

Let O be the centre of a circle with diameter AB. Draw angle $\angle ACB$ in the semicircle. This angle is the angle between the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$. If the dot product of the two vectors is zero, then $\angle C$ is a right angle.



$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (\overrightarrow{OA} - \overrightarrow{OC}) \cdot (\overrightarrow{OB} - \overrightarrow{OC})$$

OA and OB are both radii and $\overrightarrow{OB} = -\overrightarrow{OA}$.

Then
$$\begin{aligned}\overrightarrow{CA} \cdot \overrightarrow{CB} &= (\overrightarrow{OA} - \overrightarrow{OC}) \cdot (-\overrightarrow{OA} - \overrightarrow{OC}) \\ &= -\overrightarrow{OA} \cdot \overrightarrow{OA} - \overrightarrow{OA} \cdot \overrightarrow{OC} + \overrightarrow{OC} \cdot \overrightarrow{OA} + \overrightarrow{OC} \cdot \overrightarrow{OC} \\ &= -|\overrightarrow{OA}|^2 + |\overrightarrow{OC}|^2 \\ &= 0, \text{ since } |\overrightarrow{OA}| \text{ and } |\overrightarrow{OC}| \text{ are radii.}\end{aligned}$$

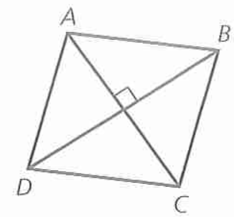
Therefore, $\angle ACB$ is a right angle.

EXAMPLE 5

If the diagonals of a parallelogram are perpendicular, prove that the parallelogram is a rhombus.

Solution

Draw and label a diagram.



Let $ABCD$ be a parallelogram. Then opposite sides are equal vectors; for instance, $\overrightarrow{AB} = \overrightarrow{DC}$. The diagonals are perpendicular, so

$$\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$$
$$(\overrightarrow{AB} + \overrightarrow{BC}) \cdot (\overrightarrow{BC} + \overrightarrow{CD}) = 0$$
$$(-\overrightarrow{CD} + \overrightarrow{BC}) \cdot (\overrightarrow{BC} + \overrightarrow{CD}) = 0$$
$$-\overrightarrow{CD} \cdot \overrightarrow{BC} - \overrightarrow{CD} \cdot \overrightarrow{CD} + \overrightarrow{BC} \cdot \overrightarrow{BC} + \overrightarrow{BC} \cdot \overrightarrow{CD} = 0$$
$$-|\overrightarrow{CD}|^2 + |\overrightarrow{BC}|^2 = 0$$

Therefore, $|\overrightarrow{BC}| = |\overrightarrow{CD}|$, so adjacent sides are equal and the figure must be a rhombus.

Exercise 6.4

Part A

Communication

1. Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length
   a. using point-to-point vectors
   b. using position vectors

Knowledge/Understanding

2. If side BC of ΔABC is trisected by points P and Q, show that $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AP} + \overrightarrow{AQ}$
   a. using point-to-point vectors
   b. using position vectors

3. If D, E, and F are the midpoints of the sides of the triangle ABC, prove that $\overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$.

Part B

4. Prove that if the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.

5. If G is the centroid of ΔABC and AD is one of its medians,
   a. in what ratio does D divide BC?
   b. in what ratio does G divide AD?
   c. Prove that $\overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$.

6. If G is the centroid of ΔABC, prove that $\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG} = \vec{0}$.

Knowledge/ Understanding

7. Prove that the diagonals of a parallelogram bisect each other. Use the type of proof shown in Example 3.

8. If a line through the centre of a circle is perpendicular to a chord, prove that it intersects the chord at its midpoint.

Application

9. Show that the midpoint of the hypotenuse of a right-angled triangle is equidistant from the vertices.

10. Prove that the sum of the squares of the diagonals of any parallelogram is equal to the sum of the squares of the four sides.

11. In the trapezoid $ABCD$, $\overrightarrow{AB} = n\overrightarrow{DC}$. If the diagonals BD and AC meet at K, show that

$$\overrightarrow{AK} = \frac{n}{n+1}\overrightarrow{AD} + \frac{1}{n+1}\overrightarrow{AB}$$

Application

12. ΔABC is inscribed in a circle with centre X. Define a point P by its position vector $\overrightarrow{XP} = \overrightarrow{XA} + \overrightarrow{XB} + \overrightarrow{XC}$.
   a. Show that $\overrightarrow{CP} = XA + XB$.
   b. Show that $\overrightarrow{CP} \perp \overrightarrow{AB}$, $\overrightarrow{BP} \perp \overrightarrow{AC}$, and $\overrightarrow{AP} \perp \overrightarrow{BC}$.
   c. Explain why the results of part **b** prove that the three altitudes of a triangle intersect at a common point. (P is known as the **orthocentre** of the triangle.)

13. Let $ABCD$ be a rectangle. Prove that
   a. $\overrightarrow{OA} \bullet \overrightarrow{OC} = \overrightarrow{OB} \bullet \overrightarrow{OD}$
   b. $|\overrightarrow{OA}|^2 + |\overrightarrow{OC}|^2 = |\overrightarrow{OB}|^2 + |\overrightarrow{OD}|^2$

Part C

14. A regular hexagon $ABCDEF$ has two of its diagonals, AC and BE, meeting at the point K. Determine the ratios in which K divides AC and BE.

Thinking/Inquiry/ Problem Solving

15. In a triangle ABC, the point E is selected on BC so that $BE:EC = 1:2$. The point F divides AC in the ratio 2:3. The two line segments BF and AE intersect at D.

    a. Find the ratios in which D divides AE and BF.

    b. Determine the ratio of the area of the quadrilateral $CEDF$ to the area of the triangle ABC.

16. In the parallelogram $ABCD$, DC is extended to E so that $DE:EC = 3:-2$. The line AE meets BC at F. Determine the ratios in which F divides BC and F divides AE.

17. In the quadrilateral $APBQ$, $|\overrightarrow{AP}| = |\overrightarrow{AQ}|$ and $|\overrightarrow{BP}| = |\overrightarrow{BQ}|$.

    a. Prove that AB bisects PQ.

    b. Prove that AB is perpendicular to PQ.

Thinking/Inquiry/ Problem Solving

18. Given the tetrahedron $MNPQ$ with $MN \perp PQ$ and $MP \perp NQ$, prove $MQ \perp NP$.