6.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, verify the solution of the differential equation.

Solution	*Differential Equation*
1. $y = Ce^{4x}$	$y' = 4y$
2. $y = e^{-2x}$	$3y' + 5y = -e^{-2x}$
3. $x^2 + y^2 = Cy$	$y' = 2xy/(x^2 - y^2)$
4. $y^2 - 2 \ln y = x^2$	$\dfrac{dy}{dx} = \dfrac{xy}{y^2 - 1}$
5. $y = C_1 \sin x - C_2 \cos x$	$y'' + y = 0$
6. $y = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x$	$y'' + 2y' + 2y = 0$
7. $y = -\cos x \ln\lvert\sec x + \tan x\rvert$	$y'' + y = \tan x$
8. $y = \frac{2}{5}(e^{-4x} + e^x)$	$y'' + 4y' = 2e^x$

In Exercises 9–12, verify the particular solution of the differential equation.

Solution	*Differential Equation and Initial Condition*
9. $y = \sin x \cos x - \cos^2 x$	$2y + y' = 2\sin(2x) - 1$, $y\left(\frac{\pi}{4}\right) = 0$
10. $y = \frac{1}{2}x^2 - 2\cos x - 3$	$y' = x + 2\sin x$, $y(0) = -5$
11. $y = 4e^{-6x^2}$	$y' = -12xy$, $y(0) = 4$
12. $y = e^{-\cos x}$	$y' = y \sin x$, $y\left(\frac{\pi}{2}\right) = 1$

In Exercises 13–20, determine whether the function is a solution of the differential equation $y^{(4)} - 16y = 0$.

13. $y = 3\cos x$
14. $y = 2\sin x$
15. $y = 3\cos 2x$
16. $y = 3\sin 2x$
17. $y = e^{-2x}$
18. $y = 5\ln x$
19. $y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \sin 2x + C_4 \cos 2x$
20. $y = 3e^{2x} - 4\sin 2x$

In Exercises 21–28, determine whether the function is a solution of the differential equation $xy' - 2y = x^3 e^x$.

21. $y = x^2$
22. $y = x^3$
23. $y = x^2 e^x$
24. $y = x^2(2 + e^x)$
25. $y = \sin x$
26. $y = \cos x$
27. $y = \ln x$
28. $y = x^2 e^x - 5x^2$

In Exercises 29–32, some of the curves corresponding to different values of C in the general solution of the differential equation are given. Find the particular solution that passes through the point shown on the graph.

Solution	*Differential Equation*
29. $y = Ce^{-x/2}$	$2y' + y = 0$
30. $y(x^2 + y) = C$	$2xy + (x^2 + 2y)y' = 0$
31. $y^2 = Cx^3$	$2xy' - 3y = 0$
32. $2x^2 - y^2 = C$	$yy' - 2x = 0$

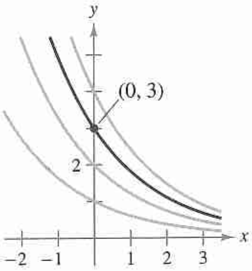


Figure for 29

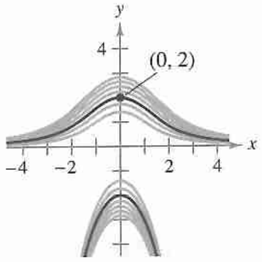


Figure for 30

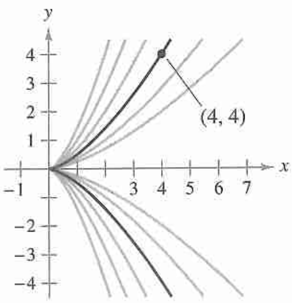


Figure for 31

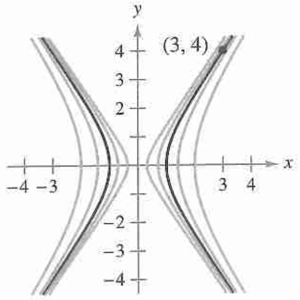


Figure for 32

In Exercises 33 and 34, the general solution of the differential equation is given. Use a graphing utility to graph the particular solutions for the given values of C.

33. $4yy' - x = 0$
 $4y^2 - x^2 = C$
 $C = 0, C = \pm 1, C = \pm 4$

34. $yy' + x = 0$
 $x^2 + y^2 = C$
 $C = 0, C = 1, C = 4$

In Exercises 35–40, verify that the general solution satisfies the differential equation. Then find the particular solution that satisfies the initial condition.

35. $y = Ce^{-2x}$
 $y' + 2y = 0$
 $y = 3$ when $x = 0$

36. $3x^2 + 2y^2 = C$
 $3x + 2yy' = 0$
 $y = 3$ when $x = 1$

37. $y = C_1 \sin 3x + C_2 \cos 3x$
 $y'' + 9y = 0$
 $y = 2$ when $x = \pi/6$
 $y' = 1$ when $x = \pi/6$

38. $y = C_1 + C_2 \ln x$
 $xy'' + y' = 0$
 $y = 0$ when $x = 2$
 $y' = \frac{1}{2}$ when $x = 2$

39. $y = C_1x + C_2x^3$

$x^2y'' - 3xy' + 3y = 0$

$y = 0$ when $x = 2$

$y' = 4$ when $x = 2$

40. $y = e^{2x/3}(C_1 + C_2x)$

$9y'' - 12y' + 4y = 0$

$y = 4$ when $x = 0$

$y = 0$ when $x = 3$

In Exercises 41–52, use integration to find a general solution of the differential equation.

41. $\dfrac{dy}{dx} = 6x^2$

42. $\dfrac{dy}{dx} = 2x^3 - 3x$

43. $\dfrac{dy}{dx} = \dfrac{x}{1 + x^2}$

44. $\dfrac{dy}{dx} = \dfrac{e^x}{4 + e^x}$

45. $\dfrac{dy}{dx} = \dfrac{x - 2}{x}$

46. $\dfrac{dy}{dx} = x \cos x^2$

47. $\dfrac{dy}{dx} = \sin 2x$

48. $\dfrac{dy}{dx} = \tan^2 x$

49. $\dfrac{dy}{dx} = x\sqrt{x - 6}$

50. $\dfrac{dy}{dx} = 2x\sqrt{3 - x}$

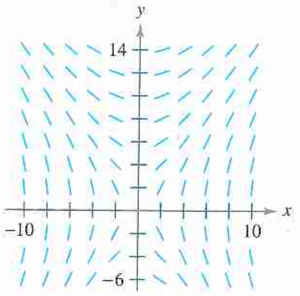
51. $\dfrac{dy}{dx} = xe^{x^2}$

52. $\dfrac{dy}{dx} = 5e^{-x/2}$

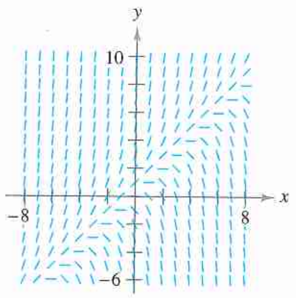
Slope Fields **In Exercises 53–56, a differential equation and its slope field are given. Complete the table by determining the slopes (if possible) in the slope field at the given points.**

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx						

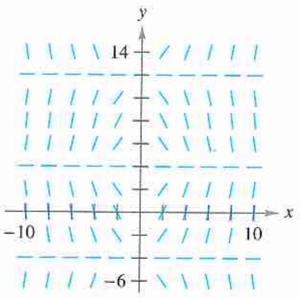
53. $\dfrac{dy}{dx} = \dfrac{2x}{y}$



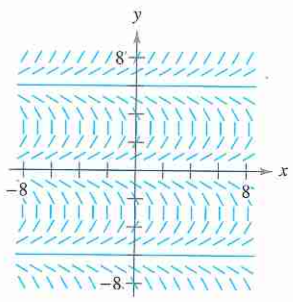
54. $\dfrac{dy}{dx} = y - x$



55. $\dfrac{dy}{dx} = x \cos \dfrac{\pi y}{8}$

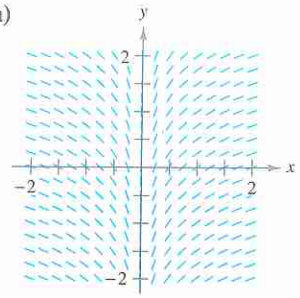


56. $\dfrac{dy}{dx} = \tan\left(\dfrac{\pi y}{6}\right)$

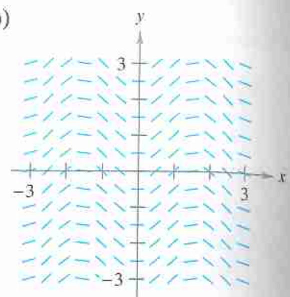


In Exercises 57–60, match the differential equation with its slope field. [The slope fields are labeled (a), (b), (c), and (d).]

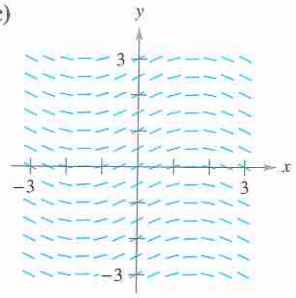
(a)



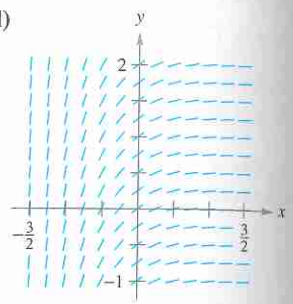
(b)



(c)



(d)



57. $\dfrac{dy}{dx} = \sin(2x)$

58. $\dfrac{dy}{dx} = \dfrac{1}{2} \cos x$

59. $\dfrac{dy}{dx} = e^{-2x}$

60. $\dfrac{dy}{dx} = \dfrac{1}{x}$

Slope Fields **In Exercises 61–64, (a) sketch the slope field for the differential equation, (b) use the slope field to sketch the solution that passes through the given point, and (c) discuss the graph of the solution as $x \to \infty$ and $x \to -\infty$. Use a graphing utility to verify your results.**

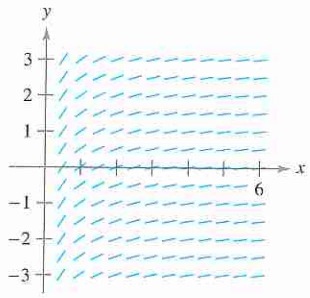
61. $y' = 3 - x, \quad (4, 2)$

62. $y' = \frac{1}{3}x^2 - \frac{1}{2}x, \quad (1, 1)$

63. $y' = y - 4x, \quad (2, 2)$

64. $y' = y + xy, \quad (0, -4)$

65. ***Slope Field*** Use the slope field for the differential equation $y' = 1/x$, where $x > 0$, to sketch the graph of the solution that satisfies each given initial condition. Then make a conjecture about the behavior of a particular solution of $y' = 1/x$ as $x \to \infty$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(a) $(1, 0)$ (b) $(2, -1)$

10.1 EXERCISES

1. Show that $y = x - x^{-1}$ is a solution of the differential equation $xy' + y = 2x$.

2. Verify that $y = \sin x \cos x - \cos x$ is a solution of the initial-value problem

$$y' + (\tan x)y = \cos^2 x \qquad y(0) = -1$$

on the interval $-\pi/2 < x < \pi/2$.

3. (a) For what values of r does the function $y = e^{rx}$ satisfy the differential equation $2y'' + y' - y = 0$?
   (b) If r_1 and r_2 are the values of r that you found in part (a), show that every member of the family of functions $y = ae^{r_1 x} + be^{r_2 x}$ is also a solution.

4. (a) For what values of k does the function $y = \cos kt$ satisfy the differential equation $4y'' = -25y$?
   (b) For those values of k, verify that every member of the family of functions $y = A \sin kt + B \cos kt$ is also a solution.

5. Which of the following functions are solutions of the differential equation $y'' + 2y' + y = 0$?
   (a) $y = e^t$ (b) $y = e^{-t}$
   (c) $y = te^{-t}$ (d) $y = t^2e^{-t}$

6. (a) Show that every member of the family of functions $y = Ce^{x^2/2}$ is a solution of the differential equation $y' = xy$.
   (b) Illustrate part (a) by graphing several members of the family of solutions on a common screen.
   (c) Find a solution of the differential equation $y' = xy$ that satisfies the initial condition $y(0) = 5$.
   (d) Find a solution of the differential equation $y' = xy$ that satisfies the initial condition $y(1) = 2$.

7. (a) What can you say about a solution of the equation $y' = -y^2$ just by looking at the differential equation?
   (b) Verify that all members of the family $y = 1/(x + C)$ are solutions of the equation in part (a).
   (c) Can you think of a solution of the differential equation $y' = -y^2$ that is not a member of the family in part (b)?
   (d) Find a solution of the initial-value problem

$$y' = -y^2 \qquad y(0) = 0.5$$

8. (a) What can you say about the graph of a solution of the equation $y' = xy^3$ when x is close to 0? What if x is large?
   (b) Verify that all members of the family $y = (c - x^2)^{-1/2}$ are solutions of the differential equation $y' = xy^3$.
   (c) Graph several members of the family of solutions on a common screen. Do the graphs confirm what you predicted in part (a)?
   (d) Find a solution of the initial-value problem

$$y' = xy^3 \qquad y(0) = 2$$

9. A population is modeled by the differential equation

$$\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$$

   (a) For what values of P is the population increasing?
   (b) For what values of P is the population decreasing?
   (c) What are the equilibrium solutions?

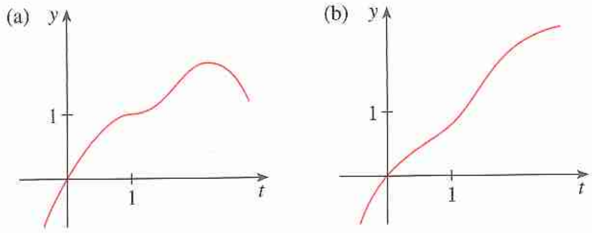
10. A function $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = y^4 - 6y^3 + 5y^2$$

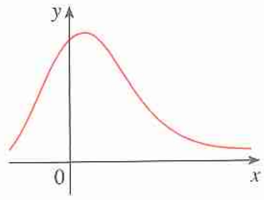
   (a) What are the constant solutions of the equation?
   (b) For what values of y is y increasing?
   (c) For what values of y is y decreasing?

11. Explain why the functions with the given graphs *can't* be solutions of the differential equation

$$\frac{dy}{dt} = e^t(y - 1)^2$$



12. The function with the given graph is a solution of one of the following differential equations. Decide which is the correct equation and justify your answer.



A. $y' = 1 + xy$ **B.** $y' = -2xy$ **C.** $y' = 1 - 2xy$

13. Psychologists interested in learning theory study **learning curves**. A learning curve is the graph of a function $P(t)$, the performance of someone learning a skill as a function of the training time t. The derivative dP/dt represents the rate at which performance improves.
   (a) When do you think P increases most rapidly? What happens to dP/dt as t increases? Explain.
   (b) If M is the maximum level of performance of which the learner is capable, explain why the differential equation

$$\frac{dP}{dt} = k(M - P) \qquad k \text{ a positive constant}$$

is a reasonable model for learning.

3 EXERCISES

10 Solve the differential equation.

$\dfrac{dy}{dx} = y^2$

2. $\dfrac{dy}{dx} = \dfrac{e^{2x}}{4y^3}$

$(x^2 + 1)y' = xy$

4. $y' = y^2 \sin x$

$(1 + \tan y)y' = x^2 + 1$

6. $\dfrac{du}{dr} = \dfrac{1 + \sqrt{r}}{1 + \sqrt{u}}$

$\dfrac{dy}{dt} = \dfrac{te^t}{y\sqrt{1 + y^2}}$

8. $\dfrac{dy}{d\theta} = \dfrac{e^y \sin^2\theta}{y \sec\theta}$

$\dfrac{du}{dt} = 2 + 2u + t + tu$

10. $\dfrac{dz}{dt} + e^{t+z} = 0$

–18 Find the solution of the differential equation that satisfies e given initial condition.

$\dfrac{dy}{dx} = y^2 + 1, \quad y(1) = 0$

$\dfrac{dy}{dx} = \dfrac{y \cos x}{1 + y^2}, \quad y(0) = 1$

$x \cos x = (2y + e^{3y})y', \quad y(0) = 0$

$\dfrac{dP}{dt} = \sqrt{Pt}, \quad P(1) = 2$

$\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}, \quad u(0) = -5$

$xy' + y = y^2, \quad y(1) = -1$

$y' \tan x = a + y, \quad y(\pi/3) = a, \quad 0 < x < \pi/2$

$\dfrac{dL}{dt} = kL^2 \ln t, \quad L(1) = -1$

. Find an equation of the curve that passes through the point $(0, 1)$ and whose slope at (x, y) is xy.

. Find the function f such that $f'(x) = f(x)(1 - f(x))$ and $f(0) = \frac{1}{2}$.

. Solve the differential equation $y' = x + y$ by making the change of variable $u = x + y$.

. Solve the differential equation $xy' = y + xe^{y/x}$ by making the change of variable $v = y/x$.

. (a) Solve the differential equation $y' = 2x\sqrt{1 - y^2}$.
(b) Solve the initial-value problem $y' = 2x\sqrt{1 - y^2}$, $y(0) = 0$, and graph the solution.
(c) Does the initial-value problem $y' = 2x\sqrt{1 - y^2}$, $y(0) = 2$, have a solution? Explain.

24. Solve the equation $e^{-y}y' + \cos x = 0$ and graph several members of the family of solutions. How does the solution curve change as the constant C varies?

CAS **25.** Solve the initial-value problem $y' = (\sin x)/\sin y$, $y(0) = \pi/2$, and graph the solution (if your CAS does implicit plots).

CAS **26.** Solve the equation $y' = x\sqrt{x^2 + 1}/(ye^y)$ and graph several members of the family of solutions (if your CAS does implicit plots). How does the solution curve change as the constant C varies?

CAS **27–28**

(a) Use a computer algebra system to draw a direction field for the differential equation. Get a printout and use it to sketch some solution curves without solving the differential equation.
(b) Solve the differential equation.
(c) Use the CAS to draw several members of the family of solutions obtained in part (b). Compare with the curves from part (a).

27. $y' = 1/y$

28. $y' = x^2/y$

29–32 Find the orthogonal trajectories of the family of curves. Use a graphing device to draw several members of each family on a common screen.

29. $x^2 + 2y^2 = k^2$

30. $y^2 = kx^3$

31. $y = \dfrac{k}{x}$

32. $y = \dfrac{x}{1 + kx}$

33. Solve the initial-value problem in Exercise 27 in Section 10.2 to find an expression for the charge at time t. Find the limiting value of the charge.

34. In Exercise 28 in Section 10.2 we discussed a differential equation that models the temperature of a 95°C cup of coffee in a 20°C room. Solve the differential equation to find an expression for the temperature of the coffee at time t.

35. In Exercise 13 in Section 10.1 we formulated a model for learning in the form of the differential equation

$$\frac{dP}{dt} = k(M - P)$$

where $P(t)$ measures the performance of someone learning a skill after a training time t, M is the maximum level of performance, and k is a positive constant. Solve this differential equation to find an expression for $P(t)$. What is the limit of this expression?

2. Which of the equations in Exercise 1 are linear?

In Exercises 3–8, verify that the given function is a solution of the differential equation.

3. $y' - 8x = 0, \quad y = 4x^2$

4. $yy' + 4x = 0, \quad y = \sqrt{12 - 4x^2}$

5. $y' + 4xy = 0, \quad y = 25e^{-2x^2}$

6. $(x^2 - 1)y' + xy = 0, \quad y = 4(x^2 - 1)^{-1/2}$

7. $y'' - 2xy' + 8y = 0, \quad y = 4x^4 - 12x^2 + 3$

8. $y'' - 2y' + 5y = 0, \quad y = e^x \sin 2x$

9. Which of the following equations are separable? Write those that are separable in the form $y' = f(x)g(y)$ (but do not solve).

(a) $xy' - 9y^2 = 0$ **(b)** $\sqrt{4 - x^2}\, y' = e^{3y} \sin x$

(c) $y' = x^2 + y^2$ **(d)** $y' = 9 - y^2$

10. The following differential equations appear similar but have very different solutions.

$$\frac{dy}{dx} = x, \qquad \frac{dy}{dx} = y$$

Solve both subject to the initial condition $y(1) = 2$.

11. Consider the differential equation $y^3 y' - 9x^2 = 0$.

(a) Write it as $y^3\, dy = 9x^2\, dx$.

(b) Integrate both sides to obtain $\frac{1}{4}y^4 = 3x^3 + C$.

(c) Verify that $y = (12x^3 + C)^{1/4}$ is the general solution.

(d) Find the particular solution satisfying $y(1) = 2$.

12. Verify that $x^2 y' + e^{-y} = 0$ is separable.

(a) Write it as $e^y\, dy = -x^{-2}\, dx$.

(b) Integrate both sides to obtain $e^y = x^{-1} + C$.

(c) Verify that $y = \ln(x^{-1} + C)$ is the general solution.

(d) Find the particular solution satisfying $y(2) = 4$.

In Exercises 13–28, use separation of variables to find the general solution.

13. $y' + 4xy^2 = 0$ **14.** $y' + x^2 y = 0$

15. $\dfrac{dy}{dt} - 20t^4 e^{-y} = 0$ **16.** $t^3 y' + 4y^2 = 0$

17. $2y' + 5y = 4$ **18.** $\dfrac{dy}{dt} = 8\sqrt{y}$

19. $\sqrt{1 - x^2}\, y' = xy$ **20.** $y' = y^2(1 - x^2)$

21. $yy' = x$ **22.** $(\ln y)y' - ty = 0$

23. $\dfrac{dx}{dt} = (t + 1)(x^2 + 1)$ **24.** $(1 + x^2)y' = x^3 y$

25. $y' = x \sec y$ **26.** $\dfrac{dy}{d\theta} = \tan y$

27. $\dfrac{dy}{dt} = y \tan t$ **28.** $\dfrac{dx}{dt} = t \tan x$

In Exercises 29–42, solve the initial value problem.

29. $y' + 2y = 0, \quad y(\ln 5) = 3$

30. $y' - 3y + 12 = 0, \quad y(2) = 1$

31. $yy' = xe^{-y^2}, \quad y(0) = -2$

32. $y^2 \dfrac{dy}{dx} = x^{-3}, \quad y(1) = 0$

33. $y' = (x - 1)(y - 2), \quad y(2) = 4$

34. $y' = (x - 1)(y - 2), \quad y(2) = 2$

35. $y' = x(y^2 + 1), \quad y(0) = 0$

36. $(1 - t)\dfrac{dy}{dt} - y = 0, \quad y(2) = -4$

37. $\dfrac{dy}{dt} = ye^{-t}, \quad y(0) = 1$

38. $\dfrac{dy}{dt} = te^{-y}, \quad y(1) = 0$

39. $t^2 \dfrac{dy}{dt} - t = 1 + y + ty, \quad y(1) = 0$

40. $\sqrt{1 - x^2}\, y' = y^2 + 1, \quad y(0) = 0$

41. $y' = \tan y, \quad y(\ln 2) = \dfrac{\pi}{2}$

42. $y' = y^2 \sin x, \quad y(\pi) = 2$

43. Find all values of a such that $y = x^a$ is a solution of

$$y'' - 12x^{-2}y = 0$$

44. Find all values of a such that $y = e^{ax}$ is a solution of

$$y'' + 4y' - 12y = 0$$

In Exercises 45 and 46, let $y(t)$ be a solution of $(\cos y + 1)\dfrac{dy}{dt} = 2t$ such that $y(2) = 0$.

45. Show that $\sin y + y = t^2 + C$. We cannot solve for y as a function of t, but, assuming that $y(2) = 0$, find the values of t at which $y(t) = \pi$.

46. Assuming that $y(6) = \pi/3$, find an equation of the tangent line to the graph of $y(t)$ at $(6, \pi/3)$.

In Exercises 47–52, use Eq. (4) and Torricelli's Law [Eq. (5)].

47. Water leaks through a hole of area $0.002\ \text{m}^2$ at the bottom of a cylindrical tank that is filled with water and has height 3 m and a base of area $10\ \text{m}^2$. How long does it take (a) for half of the water to leak out and (b) for the tank to empty?

48. At $t = 0$, a conical tank of height 300 cm and top radius 100 cm [Figure 7(A)] is filled with water. Water leaks through a hole in the bottom of area $3\ \text{cm}^2$. Let $y(t)$ be the water level at time t.

(a) Show that the tank's cross-sectional area at height y is $A(y) = \frac{\pi}{9}y^2$.

(b) Find and solve the differential equation satisfied by $y(t)$.

(c) How long does it take for the tank to empty?

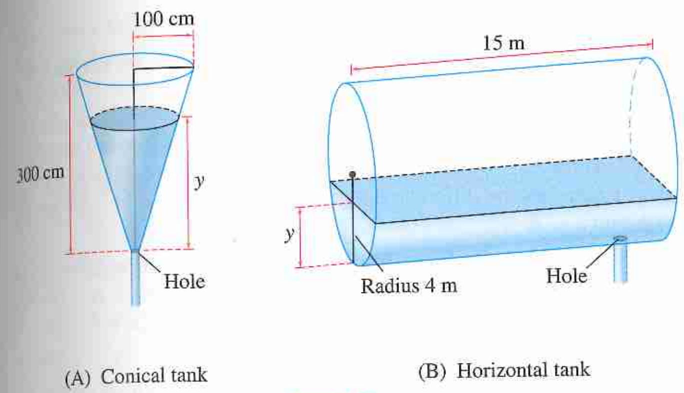


FIGURE 7

49. The tank in Figure 7(B) is a cylinder of radius 4 m and height 15 m. Assume that the tank is half-filled with water and that water leaks through a hole in the bottom of area $B = 0.001\ \text{m}^2$. Determine the water level $y(t)$ and the time t_e when the tank is empty.

50. A tank has the shape of the parabola $y = x^2$, revolved around the y-axis. Water leaks from a hole of area $B = 0.0005\ \text{m}^2$ at the bottom of the tank. Let $y(t)$ be the water level at time t. How long does it take for the tank to empty if it is initially filled to height $y_0 = 1$ m?

51. A tank has the shape of the parabola $y = ax^2$ (where a is a constant) revolved around the y-axis. Water drains from a hole of area $B\ \text{m}^2$ at the bottom of the tank.

(a) Show that the water level at time t is

$$y(t) = \left(y_0^{3/2} - \frac{3aB\sqrt{2g}}{2\pi}t\right)^{2/3}$$

where y_0 is the water level at time $t = 0$.

(b) Show that if the total volume of water in the tank has volume V at time $t = 0$, then $y_0 = \sqrt{2aV/\pi}$. *Hint:* Compute the volume of the tank as a volume of rotation.

(c) Show that the tank is empty at time

$$t_e = \left(\frac{2}{3B\sqrt{g}}\right)\left(\frac{2\pi V^3}{a}\right)^{1/4}$$

We see that for fixed initial water volume V, the time t_e is proportional to $a^{-1/4}$. A large value of a corresponds to a tall thin tank. Such a tank drains more quickly than a short wide tank of the same initial volume.

52. A cylindrical tank filled with water has height h and a base of area A. Water leaks through a hole in the bottom of area B.

(a) Show that the time required for the tank to empty is proportional to $A\sqrt{h}/B$.

(b) Show that the emptying time is proportional to $Vh^{-1/2}$, where V is the volume of the tank.

(c) Two tanks have the same volume and a hole of the same size, but they have different heights and bases. Which tank empties first: the taller or the shorter tank?

53. Figure 8 shows a circuit consisting
pacitor of C farads, and a battery of v
completed, the amount of charge $q(t)$
the capacitor varies according to the diff

$$R\frac{dq}{dt} + \frac{1}{C}q$$

where R, C, and V are constants.

(a) Solve for $q(t)$, assuming that $q(0)$

(b) Show that $\lim_{t\to\infty} q(t) = CV$.

(c) Show that the capacitor charges to
value CV after a time period of length
constant of the capacitor).

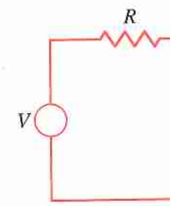


FIGURE 8 An RC

54. Assume in the circuit of Figure 8
and $V = 12$ V. How many seconds do
capacitor plates to reach half of its limi

55. According to one hypothe
cell's volume V is proportional to its su
units such as cm^3 and A has square uni
roughly that $A \propto V^{2/3}$, and hence dV
k. If this hypothesis is correct, which
would we expect to see (again, roughly

(a) Linear **(b)** Quadrat

56. We might also guess that the volu
creases at a rate proportional to its sur
55 to find a differential equation satisfi
has volume $1000\ \text{cm}^3$ and that it loses
According to this model, when will the

57. In general, $(fg)'$ is not equal to f
a function $g(x)$ such that $(fg)' = f'g$

58. A boy standing at point B on a d
tached to a boat at point A [Figure 9(
dock, holding the rope taut, the boat m
trix (from the Latin *tractus*, meaning
point P on the curve to the x-axis alo
length ℓ. Let $y = f(x)$ be the equation

(a) Show that $y^2 + (y/y')^2 = \ell^2$ an
Why must we choose the negative squ

(b) Prove that the tractrix is the graph

$$x = \ell \ln\left(\frac{\ell + \sqrt{\ell^2 -}}{y}\right.$$