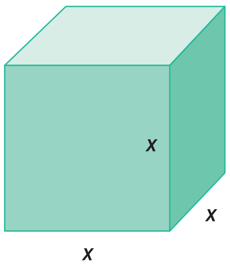
4.3 Working with Rational Exponents

GOAL

Investigate powers involving rational exponents and evaluate expressions containing them.



The volume of this cube is $V(x) = x^3$ and the area of its base is $A(x) = x^2$. In this cube, x is the side length and can be called

- the square root of A, since if squared, the result is $A(x)$
- the cube root of V, since if cubed, the result is $V(x)$

LEARN ABOUT the Math

? What exponents can be used to represent the side length x as the square root of area and the cube root of volume?

EXAMPLE 1 Representing a side length by rearranging the area formula

Express the side length x as a power of A and V.

Ira's Solution

$A = x^2$

$x = A^n$

$A = (x)(x)$

I used the area formula for the base. Since I didn't know what power to use, I used the variable n to write x as a power of A.

$A = A^n \times A^n$

$A = A^{n+n}$

$A^1 = A^{2n}$

I rewrote the area formula, substituting A^n for x.

Since I was multiplying powers with the same base, I added the exponents.

Therefore,

$1 = 2n$

I set the two exponents equal to each other. I solved this equation.

$\frac{1}{2} = n$

The exponent that represents a square root is $\frac{1}{2}$.

Therefore, $x = A^{\frac{1}{2}} = \sqrt{A}$.

EXAMPLE 2 Representing a side length by rearranging the volume formula

Sienna's Solution

$V = x^3$

$x = V^n$

$V = (x)(x)(x)$

I used the volume formula for a cube. I represented the edge length x as a power of the volume V. I used the variable n.

$V = V^n \times V^n \times V^n$

I rewrote the volume formula, substituting V^n for x.

$V = V^{n+n+n}$

I added the exponents.

$V^1 = V^{3n}$

Therefore,

$1 = 3n$

I set the two exponents equal to each other. I solved this equation.

$\frac{1}{3} = n$

The exponent that represents a cube root is $\frac{1}{3}$.

Therefore, $x = V^{\frac{1}{3}} = \sqrt[3]{V}$.

Reflecting

A. Why could x be expressed as both a square root and a cube root?

B. Make a conjecture about the meaning of $x^{\frac{1}{n}}$. Explain your reasoning.

C. Do the rules for multiplying powers with the same base still apply if the exponents are rational numbers? Create examples to illustrate your answer.

APPLY the Math

EXAMPLE 3 Connecting radical notation and exponents

Express the following in radical notation. Then evaluate.

a) $49^{-\frac{1}{2}}$ **b)** $(-8)^{\frac{1}{3}}$ **c)** $10\ 000^{\frac{1}{4}}$

Donato's Solution

a) $49^{-\frac{1}{2}} = \frac{1}{49^{\frac{1}{2}}}$

$= \frac{1}{\sqrt{49}}$

$= \frac{1}{7}$

I wrote the power using the reciprocal of its base and its opposite exponent. An exponent of $\frac{1}{2}$ means square root.

I evaluated the power.

index (plural indices)

the number at the left of the radical sign. It tells which root is indicated: 3 for cube root, 4 for fourth root, etc. If there is no number, the square root is intended.

b) $(-8)^{\frac{1}{3}} = \sqrt[3]{-8}$
$= -2$

An exponent of $\frac{1}{3}$ means cube root. I wrote the root as a radical, using an index of 3. That means the number is multiplied by itself three times to get -8. The number is -2.

c) $10\,000^{\frac{1}{4}} = \sqrt[4]{10\,000}$
$= 10$

An exponent of $\frac{1}{4}$ means the fourth root, since
$10\,000^{\frac{1}{4}} \times 10\,000^{\frac{1}{4}} \times$
$10\,000^{\frac{1}{4}} \times 10\,000^{\frac{1}{4}} = 10\,000^{1}$.

That number must be 10.

EXAMPLE 4 Selecting an approach to evaluate a power

Evaluate $27^{\frac{2}{3}}$.

Cory's Solutions

$27^{\frac{2}{3}}$

I know that the exponent $\frac{1}{3}$ indicates a cube root. So I used the power-of-a-power rule to separate the exponents:

$\frac{2}{3} = 2 \times \frac{1}{3}$ and $\frac{2}{3} = \frac{1}{3} \times 2$

$$= 27^{\frac{1}{3}\times 2} \qquad = 27^{2\times\frac{1}{3}}$$
$$= (27^{\frac{1}{3}})^2 \qquad = (27^2)^{\frac{1}{3}}$$
$$= (\sqrt[3]{27})^2 \qquad = \sqrt[3]{27^2}$$
$$= (3)^2 \qquad = \sqrt[3]{729}$$
$$= 9 \qquad = 9$$

To see if the order in which I applied the exponents mattered, I calculated the solution in two ways.

In the first way, I evaluated the cube root before squaring the result.

In the other way, I squared the base and then took the cube root of the result.

Both ways resulted in 9.

EXAMPLE 5 Evaluating a power with a rational exponent

Evaluate.

a) $(-27)^{\frac{4}{3}}$ **b)** $(16)^{-0.75}$

Casey's Solutions

a) $(-27)^{\frac{4}{3}} = ((-27)^{\frac{1}{3}})^4$
$= (\sqrt[3]{-27})^4$
$= (-3)^4$
$= 81$

I rewrote the exponent as $4 \times \frac{1}{3}$.
I represented $(-27)^{\frac{1}{3}}$ as $\sqrt[3]{-27}$.
I calculated the cube root of -27.
I evaluated the power.

b) $16^{-0.75} = 16^{-\frac{3}{4}}$
$= \frac{1}{16^{\frac{3}{4}}}$
$= \frac{1}{(\sqrt[4]{64})^3}$
$= \frac{1}{2^3}$
$= \frac{1}{8}$

I rewrote the power, changing the exponent from -0.75 to its equivalent fraction.

I expressed $16^{-\frac{3}{4}}$ as a rational number, using 1 as the numerator and $16^{\frac{3}{4}}$ as the denominator.

I determined the fourth root of 64 and cubed the result.

The rules of exponents also apply to powers involving rational exponents.

EXAMPLE 6 Representing an expression involving the same base as a single power

Simplify, and then evaluate $\frac{8^{\frac{5}{6}}\sqrt{8}}{8^{\frac{5}{3}}}$.

Lucia's Solution

$\frac{8^{\frac{5}{6}}\sqrt{8}}{8^{\frac{5}{3}}} = \frac{8^{\frac{5}{6}}8^{\frac{1}{2}}}{8^{\frac{5}{3}}}$
$= \frac{8^{\frac{5}{6}+\frac{1}{2}}}{8^{\frac{5}{3}}}$

To simplify, I converted the radical into exponent form.

Since the bases were the same, I wrote the numerator as a single power by adding exponents, then I subtracted exponents to simplify the whole expression.

$$= \frac{8^{\frac{4}{3}}}{8^{\frac{5}{3}}}$$

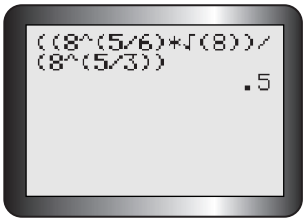
$$= 8^{\frac{4}{3} - \frac{5}{3}}$$

$$= 8^{-\frac{1}{3}}$$

Once I had simplified to a single power of 8, the number was easier to evaluate.

$$= \frac{1}{8^{\frac{1}{3}}}$$

$$= \frac{1}{2}$$



I checked my work on my calculator.

In Summary

Key Ideas

- A number raised to a rational exponent is equivalent to a radical. The rational exponent $\frac{1}{n}$ indicates the nth root of the base. If $n > 1$ and $n \in N$, then $b^{\frac{1}{n}} = \sqrt[n]{b}$, where $b \neq 0$.
- If the numerator of a rational exponent is not 1, and if m and n are positive integers, then $b^{\frac{m}{n}} = (\sqrt[n]{b})^m = \sqrt[n]{b^m}$, where $b \neq 0$.

Need to Know

- The exponent laws that apply to powers with integer exponents also apply to powers with rational exponents. Included are the product-of-powers rule $a^n \times b^n = (ab)^n$ and the quotient of powers rule $a^n \div b^n = \left(\frac{a}{b}\right)^n$.
- The power button on a scientific calculator can be used to evaluate rational exponents.
- Some roots of negative numbers do not have real solutions. For example, -16 does not have a real-number square root, since whether you square a positive or negative number, the result is positive.
- Odd roots can have negative bases, but even ones cannot.

CHECK Your Understanding

1. Write in radical form. Then evaluate without using a calculator.

a) $49^{\frac{1}{2}}$ c) $(-125)^{\frac{1}{3}}$ e) $81^{\frac{1}{4}}$

b) $100^{\frac{1}{2}}$ d) $16^{0.25}$ f) $-(144)^{0.5}$

2. Write in exponent form, then evaluate. Express answers in rational form.

a) $\sqrt[9]{512}$ c) $\sqrt[3]{27^2}$ e) $\sqrt[5]{\frac{-32}{243}}$

b) $\sqrt[3]{-27}$ d) $(\sqrt[3]{-216})^5$ f) $\sqrt[4]{\left(\frac{16}{81}\right)^{-1}}$

3. Write as a single power.

a) $8^{\frac{2}{3}}(8^{\frac{1}{3}})$ d) $(7^{\frac{5}{6}})^{-\frac{6}{5}}$

b) $8^{\frac{2}{3}} \div 8^{\frac{1}{3}}$ e) $\frac{9^{\frac{-1}{5}}}{9^{\frac{2}{3}}}$

c) $(-11)^2(-11)^{\frac{3}{4}}$ f) $10^{-\frac{4}{5}}(10^{\frac{1}{15}}) \div 10^{\frac{2}{3}}$

PRACTISING

4. Write as a single power, then evaluate. Express answers in rational form.

a) $\sqrt{5}\sqrt{5}$ b) $\frac{\sqrt[3]{-16}}{\sqrt[3]{2}}$ c) $\frac{\sqrt{28}\sqrt{4}}{\sqrt{7}}$ d) $\frac{\sqrt[4]{18}\,(\sqrt[4]{9})}{\sqrt[4]{2}}$

5. Evaluate.

a) $49^{\frac{1}{2}} + 16^{\frac{1}{2}}$ d) $128^{-\frac{5}{7}} - 16^{0.75}$

b) $27^{\frac{2}{3}} - 81^{\frac{3}{4}}$ e) $16^{\frac{3}{2}} + 16^{-0.5} + 8 - 27^{\frac{2}{3}}$

c) $16^{\frac{3}{4}} + 16^{\frac{3}{4}} - 81^{-\frac{1}{4}}$ f) $81^{\frac{1}{2}} + \sqrt[3]{8} - 32^{\frac{4}{5}} + 16^{\frac{3}{4}}$

6. Write as a single power, then evaluate. Express answers in rational form.

a) $4^{\frac{1}{5}}(4^{0.3})$ c) $\frac{64^{\frac{4}{3}}}{64}$ e) $\frac{(16^{-2.5})^{-0.2}}{16^{\frac{3}{4}}}$

b) $100^{0.2}(100^{\frac{-7}{10}})$ d) $\frac{27^{-1}}{27^{\frac{-2}{3}}}$ f) $\frac{(8^{-2})(8^{2.5})}{(8^6)^{-0.25}}$

7. Predict the order of these six expressions in terms of value from lowest to highest. Check your answers with your calculator. Express answers to three decimal places.

a) $\sqrt[4]{623}$ c) $\sqrt[10]{10.24}$ e) $17.5^{\frac{5}{8}}$

b) $125^{\frac{2}{5}}$ d) $80.9^{\frac{1}{4}}$ f) $21.4^{\frac{3}{2}}$

8. The volume of a cube is 0.015 625 m^3. Determine the length of each side.

A

9. Use your calculator to determine the values of $27^{\frac{4}{3}}$ and $27^{1.3333}$. Compare the two answers. What do you notice?

10. Explain why $(-100)^{0.2}$ is possible to evaluate while $(-100)^{0.5}$ is not.

C

11. Write $125^{\frac{-2}{3}}$ in radical form, then evaluate. Explain each of your steps.

K

12. Evaluate.

a) $-256^{0.375}$ c) $\sqrt[3]{-0.027^4}$ e) $\sqrt[4]{(0.0016)^3}$

b) $15.625^{\frac{4}{3}}$ d) $(-3.375)^{\frac{2}{3}}$ f) $(-7776)^{1.6}$

13. The power 4^3 means that 4 is multiplied by itself three times. Explain the meaning of $4^{2.5}$.

14. State whether each expression is true or false.

a) $9^{\frac{1}{2}} + 4^{\frac{1}{2}} = (9 + 4)^{\frac{1}{2}}$ d) $\left(\frac{1}{a} \times \frac{1}{b}\right)^{-1} = ab$

b) $9^{\frac{1}{2}} + 4^{\frac{1}{2}} = (9 \times 4)^{\frac{1}{2}}$ e) $\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^6 = x^2 + y^2$

c) $\left(\frac{1}{a} + \frac{1}{b}\right)^{-1} = a + b$ f) $\left[\left(x^{\frac{1}{3}}\right)\left(y^{\frac{1}{3}}\right)\right]^6 = x^2y^2$

15. a) What are some values of m and n that would make $(-2)^{\frac{m}{n}}$ undefined?

T b) What are some values of m and n that would make $(6)^{\frac{m}{n}}$ undefined?

Extending

16. Given that $x^y = y^x$, what could x and y be? Is there a way to find the answer graphically?

17. Mary must solve the equation $1.225 = (1 + i)^{12}$ to determine the value of each dollar she invested for a year at the interest rate i per year. Her friend Bindu suggests that she begin by taking the 12th root of each side of the equation. Will this work? Try it and solve for the variable i. Explain why it does or does not work.

18. Solve.

a) $\left(\frac{1}{16}\right)^{\frac{1}{4}} - \sqrt[3]{\frac{8}{27}} = \sqrt{x^2}$

b) $\sqrt[3]{\frac{1}{8}} - \sqrt[4]{x^4} + 15 = \sqrt[4]{16}$