1.5 Factoring Polynomials

SETTING THE STAGE

To explore the concepts in this lesson in more detail, see Exploration 1 on page 559.

To sketch the graph of a polynomial function you need to know how the function behaves for large and small values of x. You have seen that this behaviour is influenced by the degree of the function as well as by the leading coefficient. The zeros of the function also define the shape of the curve. To determine the zeros of a polynomial function, convert the function from standard form to factored form. In this section, you will factor polynomials with a degree of 3 and higher.

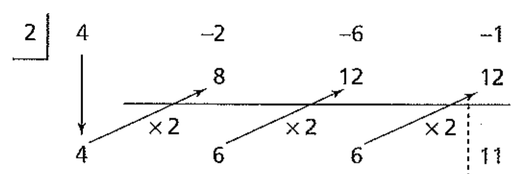
EXAMINING THE CONCEPT

Developing the Remainder Theorem

To evaluate the polynomial function $P(x) = 4x^3 - 2x^2 - 6x - 1$ for $x = 2$, substitute 2 for x.

$$\begin{aligned} P(2) &= 4(2)^3 - 2(2)^2 - 6(2) - 1 \\ &= 32 - 8 - 12 - 1 \\ &= 11 \end{aligned}$$

To find the remainder when $P(x)$ is divided by $x - 2$, you can use synthetic division:



When $P(x)$ is divided by $x - 2$, the remainder is equal to $P(2)$. This result is not coincidental. Use the division statement to verify why this result occurs.

$$\begin{aligned} \text{dividend} &= \text{quotient} \times \text{divisor} + \text{remainder} \\ P(x) &= (4x^2 + 6x + 6)(x - 2) + 11 \\ P(2) &= [4(2)^2 + 6(2) + 6](2 - 2) + 11 \\ &= (34)(0) + 11 \\ &= 0 + 11 \\ &= 11 \end{aligned}$$

When the dividend is evaluated at the value that makes the binomial divisor equal to 0, the remainder is the end result.

Case 1: $P(x) \div (x - k)$

$P(k) = r$, if $Q(x)$ is the quotient and r is the remainder when the polynomial $P(x)$ is divided by $x - k$.

$$\begin{aligned} P(x) &= Q(x) \times (x - k) + r \\ P(k) &= Q(k) \times (k - k) + r \\ &= Q(k) \times 0 + r \\ &= 0 + r \\ &= r \end{aligned}$$

Case 2: $P(x) \div (jx - k)$

$P\left(\frac{k}{j}\right) = r$, if $Q(x)$ is the quotient and r is the remainder when the polynomial $P(x)$ is divided by $jx - k$.

$$\begin{aligned} P(x) &= Q(x) \times (jx - k) + r \\ P\left(\frac{k}{j}\right) &= Q\left(\frac{k}{j}\right) \times \left[j\left(\frac{k}{j}\right) - k\right] + r \\ &= Q\left(\frac{k}{j}\right) \times (k - k) + r \\ &= Q\left(\frac{k}{j}\right) \times 0 + r \\ &= 0 + r \\ &= r \end{aligned}$$

The Remainder Theorem

When a polynomial function $P(x)$ is divided by $x - k$, the remainder, r, is $P(k)$.

When a polynomial function $P(x)$ is divided by $jx - k$, the remainder, r, is $P\left(\frac{k}{j}\right)$.

Example 1 **Using the Remainder Theorem**

What is the remainder when $x^3 + 6x^2 - x - 30$ is divided by $x + 5$?

Solution

Use the **remainder theorem**. When $x + 5 = 0$, $x = -5$. The remainder is $P(-5)$.

$$\begin{aligned} P(x) &= x^3 + 6x^2 - x - 30 \\ P(-5) &= (-5)^3 + 6(-5)^2 - (-5) - 30 \\ &= -125 + 150 + 5 - 30 \\ &= 0 \end{aligned}$$

In the previous example, the remainder is 0, which means that $x + 5$ divides evenly into the polynomial $P(x)$. In general, when $P(k) = 0$, then $x - k$ is a factor of $P(x)$. Use this relation to determine the factors of any polynomial.

The Factor Theorem

A polynomial function $P(x)$ has a factor $x - k$ if and only if $P(k) = 0$.
Therefore, if $P(k) = 0$, then $x - k$ is a factor.
If $x - k$ is a factor, then $P(k) = 0$.
Similarly, $jx - k$ is a factor of $P(x)$ if and only if $P\left(\frac{k}{j}\right) = 0$.

Example 2 **Using the Factor Theorem**

Which binomials are factors of $P(x) = 2x^3 - x^2 - 7x + 6$?

(a) $x + 3$ **(b)** $2x - 3$

Solution

(a) Evaluate $P(x)$ when $x + 3 = 0$ or $x = -3$.

$$\begin{aligned} P(-3) &= 2(-3)^3 - (-3)^2 - 7(-3) + 6 \\ &= -54 - 9 + 21 + 6 \\ &= -36 \end{aligned}$$

Because $P(-3) \neq 0$, $x + 3$ is not a factor of $P(x)$.

(b) Evaluate $P(x)$ when $2x - 3 = 0$ or $x = \frac{3}{2}$.

$$\begin{aligned} P\left(\frac{3}{2}\right) &= 2\left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^2 - 7\left(\frac{3}{2}\right) + 6 \\ &= 2\left(\frac{27}{8}\right) - \frac{9}{4} - \frac{21}{2} + 6 \\ &= \frac{27}{4} - \frac{9}{4} - \frac{42}{4} + \frac{24}{4} \\ &= \frac{0}{4} \\ &= 0 \end{aligned}$$

Because $P\left(\frac{3}{2}\right) = 0$, $2x - 3$ is a factor of $P(x)$.

Example 3 **Factoring a Polynomial Using the Factor Theorem**

Factor $x^3 + 2x^2 - 11x - 12$.

Solution

Use the factor theorem to find a binomial factor of the form $x - k$. Systematically substitute various values of k into the polynomial until a substitution results in 0. Use values for k that are factors of the constant term, -12. In this case, the factors of -12 are ±1, ±2, ±3, ±4, ±6, and ±12.

When $k = 1$,

$$\begin{aligned} f(1) &= (1)^3 + 2(1)^2 - 11(1) - 12 \\ &= 1 + 2 - 11 - 12 \\ &= -20 \end{aligned}$$

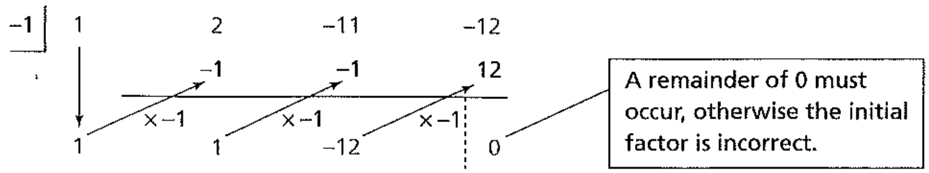
$\therefore x - 1$ is not a factor.

When $k = -1$,

$$\begin{aligned} f(-1) &= (-1)^3 + 2(-1)^2 - 11(-1) - 12 \\ &= -1 + 2 + 11 - 12 \\ &= 0 \end{aligned}$$

$\therefore x + 1$ is a factor.

Use synthetic division to divide $x^3 + 2x^2 - 11x - 12$ by $x + 1$ to determine another factor.



Therefore,

$$\begin{aligned} x^3 + 2x^2 - 11x - 12 &= (x + 1)(x^2 + x - 12) \quad \text{Factor the remaining trinomial.} \\ &= (x + 1)(x + 4)(x - 3) \end{aligned}$$

In any polynomial that can be factored, the factors are of the form $(x - k)$ or $(jx - k)$. As a result, only a rational number of the form $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient, is a factor of the polynomial.

The Rational Zero Test

In the polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, all coefficients are integers. Every rational zero of $P(x)$ is of the form $\frac{p}{q}$, where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

Example 4 Sketching the Graph of a Polynomial Function in Standard Form

(a) Graph $f(x) = 3x^3 + x^2 - 22x - 24$.

(b) Describe the shape of the graph.

Solution

(a) Begin by determining the zeros of $f(x)$. Express the function in factored form. Numbers that could make $f(x) = 0$ are of the form $\frac{p}{q}$, where p is a factor of -24 and q is a factor of 3.

$p \in \{\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24\}$, $q \in \{\pm1, \pm3\}$, and

$\frac{p}{q} \in \left\{\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}\right\}$

Systematically substitute the numbers for $\frac{p}{q}$ into $f(x)$ until one number produces a value of 0.

$$\begin{aligned} f(1) &= 3(1)^3 + (1)^2 - 22(1) - 24 \\ &= -42 \end{aligned}$$

$\therefore x - 1$ is not a factor.

$$\begin{aligned} f(-1) &= 3(-1)^3 + (-1)^2 - 22(-1) - 24 \\ &= -4 \end{aligned}$$

$\therefore x + 1$ is not a factor.

$$\begin{aligned} f(2) &= 3(2)^3 + (2)^2 - 22(2) - 24 \\ &= -40 \end{aligned}$$

$\therefore x - 2$ is not a factor.

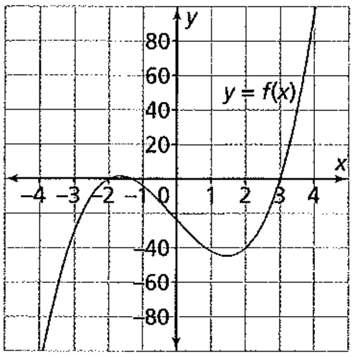
$$\begin{aligned} f(-2) &= 3(-2)^3 + (-2)^2 - 22(-2) - 24 \\ &= 0 \end{aligned}$$

$\therefore x + 2$ is a factor.

Use synthetic division to determine a second factor.

−2	3	1	−22	−24
		−6	10	24
		× −2	× −2	× −2
	3	−5	−12	0

$$
\begin{aligned}
f(x) &= 3x^3 + x^2 - 22x - 24 \\
&= (x + 2)(3x^2 - 5x - 12) \quad \text{Factor the trinomial.} \\
&= (x + 2)(3x + 4)(x - 3)
\end{aligned}
$$



$f(x) = 3x^3 + x^2 - 22x - 24$

Therefore, $f(x)$ has zeros -2, $-\frac{4}{3}$, and 3. Find additional points using a table, and determine the end behaviour of the function.

x	−3	−1	0	1	2	4
$f(x)$	−30	−4	−24	−42	−40	96

$f(x)$ has an odd degree and a positive leading coefficient.

As $x \to \infty, f(x) \to \infty$.

As $x \to -\infty, f(x) \to -\infty$.

(b) From this graph, $f(x)$ increases to a local maximum at about $(-1.7, 1.5)$. $f(x)$ then decreases to a local minimum at about $(1.5, -45)$. $f(x)$ then increases indefinitely.

Example 5 **Factoring Polynomials by Grouping**

Factor $x^4 - 6x^3 + 2x^2 - 12x$.

Solution

You can usually use the factor theorem to factor a polynomial. But first check that the terms can be grouped and whether any common factors can be removed.

$$
\begin{aligned}
x^4 - 6x^3 + 2x^2 - 12x &= (x^4 - 6x^3) + (2x^2 - 12x) && \text{Group first two and last two terms. Factor out } x^3 \text{ and } 2x. \\
&= x^3(x - 6) + 2x(x - 6) && \text{Factor out } (x - 6). \\
&= (x - 6)(x^3 + 2x) && \text{Factor out } x. \\
&= x(x - 6)(x^2 + 2)
\end{aligned}
$$

CHECK, CONSOLIDATE, COMMUNICATE

1. Explain how to determine, without actually dividing, the remainder when $3x^4 - 2x^3 + 4x^2 - x + 7$ is divided by $x - 3$.
2. Explain the two different methods for determining whether $x + 6$ is a factor of $6x^3 + 47x^2 + 59x - 42$.
3. Let $f(x) = 2x^3 + 13x^2 + 5x - 6$. List all rational numbers that could be substituted into $f(x)$ when determining whether $f(x) = 0$.

KEY IDEAS

- Remainder Theorem: When polynomial function $P(x)$ is divided by $x - k$, the remainder, r, is $P(k)$. When polynomial function $P(x)$ is divided by $jx - k$, the remainder, r, is $P\left(\frac{k}{j}\right)$.
- Factor Theorem: Polynomial function $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$. So, if $f(k) = 0$, $x - k$ is a factor, and if $x - k$ is a factor, $f(k) = 0$. Similarly, polynomial function $f(x)$ has a factor $jx - k$ if and only if $f\left(\frac{k}{j}\right) = 0$.
- To factor a polynomial of degree 3 or higher:
  1. Try to factor the polynomial by grouping.
  2. If grouping fails, find one factor by substituting numbers (according to the rational zero test) into the polynomial until one number produces a value of 0.
  3. Use either long or synthetic division to find another factor.
  4. If the resulting factor is greater than degree 2, repeat steps 1 and 2. Otherwise, go to step 4.
  5. If one factor is a quadratic expression, factor it, if possible.

1.5 Exercises

A

1. Divide to show that $x - 3$ is a factor of each polynomial. Confirm your answers using the factor theorem.

(a) $x^2 + x - 12$

(b) $x^3 - 13x + 12$

(c) $-2x^4 - 7x^3 + 22x^2 + 63x - 36$

(d) $x^4 - 12x^3 + 54x^2 - 108x + 81$

(e) $6x^4 + 13x^3 - 89x^2 - 17x + 15$

(f) $x^5 + 2x^4 - 18x^3 - 36x^2 + 81x + 162$

2. State the remainder when $x + 2$ is divided into each polynomial.

(a) $x^2 + 7x + 9$

(b) $6x^3 + 19x^2 + 11x - 11$

(c) $x^4 - 5x^2 + 4$

(d) $x^4 - 2x^3 - 11x^2 + 10x - 2$

(e) $x^3 + 3x^2 - 10x + 6$

(f) $4x^4 + 12x^3 - 13x^2 - 33x + 18$

3. Determine whether $2x - 5$ is a factor of each polynomial.

(a) $2x^3 - 5x^2 - 2x + 5$

(b) $3x^3 + 2x^2 - 3x - 2$

(c) $2x^4 - 7x^3 - 13x^2 + 63x - 45$

(d) $6x^4 + x^3 - 7x^2 - x + 1$

4. Knowledge and Understanding: Which expression is a factor of $8x^3 - 125$: $(3x + 2)$, $(x - 5)$, or $(2x - 5)$? Justify your decision.

5. Factor using the factor theorem.

(a) $x^3 - 3x^2 - 10x + 24$
(b) $4x^3 + 12x^2 - x - 15$
(c) $x^4 + 8x^3 + 4x^2 - 48x$
(d) $4x^4 + 7x^3 - 80x^2 - 21x + 270$
(e) $x^5 - 5x^4 - 7x^3 + 29x^2 + 30x$
(f) $x^4 + 2x^3 - 23x^2 - 24x + 144$

6. Factor fully.

(a) $x^3 + 9x^2 + 8x - 60$
(b) $x^3 - 7x - 6$
(c) $x^4 - 5x^2 + 4$
(d) $x^4 + 3x^3 - 38x^2 + 24x + 64$
(e) $x^3 - x^2 + x - 1$
(f) $x^5 - x^4 + 2x^3 - 2x^2 + x - 1$

B

7. Communication: Suppose that $f(x)$ is a cubic polynomial function with integral coefficients. Describe how you could find a zero and use it to find any other zeros that $f(x)$ might have.

8. Determine whether or not the given value is a zero of $f(x)$. If the value is a zero, determine any other zeros the function might have.

(a) $f(x) = 2x^3 + x^2 - 13x + 6, x = 0.5$
(b) $f(x) = 6x^3 + 17x^2 - 4x - 3, x = -2.25$
(c) $f(x) = x^3 - 2x^2 - 21x - 18, x = 6$
(d) $f(x) = 2x^4 - x^3 - 26x^2 - 11x + 12, x = 4$
(e) $f(x) = x^4 - 3x^3 + 3x^2 - 3x + 2, x = 2$
(f) $f(x) = 3x^4 - 2x^3 + 5x^2 - 2x + 7, x = -3$

9. Factor using the factor theorem.

(a) $6x^3 + 5x^2 - 21x + 10$
(b) $9x^3 - 3x^2 - 41x + 35$
(c) $6x^4 - 19x^3 - 2x^2 + 44x - 24$
(d) $10x^4 + 13x^3 - 43x^2 - 52x + 12$
(e) $8x^3 + 12x^2 - 2x - 3$
(f) $30x^3 - x^2 - 6x + 1$

10. (a) Use the factor theorem to factor.

i. $x^3 - 1$ **ii.** $x^3 - 27$ **iii.** $x^3 - 125$ **iv.** $8x^3 - 27$

(b) A polynomial in the form $a^3 - b^3$ is called a **difference of cubes.** Use the pattern of factoring in (a) to factor $a^3 - b^3$.

(c) Use the result of (b) to factor $64x^3 - 27$. Check by multiplying the factors.

11. (a) Use the factor theorem to factor.

i. $x^3 + 1$ **ii.** $x^3 + 8$ **iii.** $x^3 + 64$ **iv.** $8x^3 + 125$

(b) A polynomial in the form $a^3 + b^3$ is called a **sum of cubes.** Use the pattern of factoring in (a) to factor $a^3 + b^3$.

(c) Use the result of (b) to factor $125x^3 + 64$. Check by multiplying the factors.

12. Factor fully.

(a) $x^3 - 64$ (b) $y^3 + 27$

(c) $64x^3 + 1$ (d) $125x^6 - 27$

(e) $8x^9 + 216$ (f) $512 - 27x^{12}$

(g) $x^6 + 1331$ (h) $343x^{15} - 8$

13. Factor by grouping.

(a) $2x^3 + 2x^2 + x + 1$ (b) $3x^3 + 6x^2 - x - 2$

(c) $8x^3 + 12x^2 + 2x + 3$ (d) $10x^3 + 5x^2 - 4x - 2$

(e) $6x^5 - 2x^4 - 9x^2 + 3x$ (f) $6x^6 + 9x^5 - 4x - 6$

14. Graph $f(x) = 2x^3 - 3x^2 - 3x + 2$ using x-intercepts, the end behaviour of $f(x)$, and selected points on the graph.

15. Verify your answer to question 14 using graphing technology.

16. Graph $f(x) = -6x^4 - 23x^3 - 23x^2 + 2x + 8$ using x-intercepts, the end behaviour of $f(x)$, and selected points on the graph.

17. Verify your answer to question 16 using graphing technology.

18. The polynomial $12x^3 + kx^2 - x - 6$ has factors $2x - 1$ and $2x + 3$. Determine the value of k.

19. **Application:** When $ax^3 - x^2 + 2x + b$ is divided by $x - 1$, the remainder is 10. When it is divided by $x - 2$, the remainder is 51. Find a and b.

20. The volume of a box is $V(x) = x^3 - 15x^2 + 66x - 80$.

(a) Determine expressions for the dimensions of the box in terms of x.

(b) Graph the volume function and indicate any restrictions on x.

(c) Explain why $x = 7$ is inadmissible in the context of the question.

21. **Thinking, Inquiry, Problem Solving:** Determine a general rule to help decide whether $(x - a)$ and $(x + a)$ are factors of $x^n - a^n$ and $x^n + a^n$.

22. **Check Your Understanding:** Determine the factors of $f(x) = 2x^4 - x^3 - 14x^2 - 5x + 6$.

23. Verify your answer to question 22 using graphing technology.

24. The graph of $f(x) = ax^4 + bx^2 + cx - 24$ crosses the x-axis at 1, -2, and 3.

(a) Determine all zeros of $f(x)$.

(b) Graph $f(x)$ and comment on its end behaviour.

(c) State the coordinates of all turning points and indicate whether each turning point is a local maximum or local minimum for $f(x)$.

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

Knowledge and Understanding: For $f(x) = x^3 + 16 - 4x - 4x^2$, find the zeros and end behaviour of f. Use this and any other information to graph f by hand.

Application: The volume of a rectangular-based prism is $V(x) = -8x + x^3 - 5x^2 + 12$.

(a) Express the height, width, and depth of the prism in terms of x.

(b) Indicate any restrictions for x. Justify your restrictions.

Thinking, Inquiry, Problem Solving: Show in more than one way that $f(x) = x^6 + x^4 + x^2 + 4$ has no factors in the form $x - k$, where k is any real number.

Communication: Let $f(x) = x^3 + 8x^2 - 11x - 15$ and $g(x) = 4x^3 + 8x^2 - 11x - 15$. Send an e-mail to a classmate listing possible binomial factors for $f(x)$, and explain how you determined them. Explain why the possible binomial factors for $g(x)$ would be different. List four possible binomial factors for $g(x)$ that are not in the list of $f(x)$.

The Chapter Problem

Developing a Model for Canada's Population

In this section, you learned that $P(k) = 0$ if $x = k$ is a zero of a polynomial function, $P(x)$, and $x - k$ is a factor of $P(x)$. Apply what you learned to answer these questions about The Chapter Problem on page 2.

CP13. Using graphing technology, create a scatter plot. Enter the years since 1851 in **L1** and the population in **L2**. Using regression, determine a degree-3 polynomial model and a degree-4 polynomial model. Which model fits the data better?

CP14. What are the end behaviours of each model? Which function's end behaviour, as $x \to \infty$, better reflects the trend in the actual data?

CP15. Verify that $x - 40$ and $x - 110$ are not factors of the quartic regression equation.