1.4 Dividing Polynomials

SETTING THE STAGE

You can use the zeros of a polynomial function to sketch the graph of the function or to determine the restricted domain. In both cases, you can easily find the zeros if the function is in factored form. So you must be able to factor polynomial functions, which involves dividing one polynomial by another. In this section, you will examine techniques for dividing polynomials.

EXAMINING THE CONCEPT

Long Division with Polynomials

Dividing polynomials is similar to dividing numbers using long division. Evaluate $17 \div 4$.

Think $\frac{17}{4} \doteq 4$

$$\begin{array}{rrl} & 4 & \leftarrow \text{quotient} \\ \text{divisor} \rightarrow 4 & \overline{)17} & \leftarrow \text{dividend} \\ & \underline{-16} & \leftarrow \text{subtract } 4 \times 4 \\ & 1 & \leftarrow \text{remainder} \end{array}$$

Therefore, $17 \div 4 = 4\frac{1}{4}$. Check: $4 \times 4 + 1 = 17$

Division Statement

In any division, divisor × quotient + remainder = dividend.

Keep the division statement in mind when dividing polynomials.

Example 1 **Long Division with Polynomials**

Divide $x^2 + 3x - 28$ by $x + 5$.

Solution

Think $\frac{x^2}{x} = x$.

Think $-\frac{2x}{x} = -2$.

$$\begin{array}{rll} & x - 2 & \\ x + 5 & \overline{)x^2 + 3x - 28} & \\ & \underline{x^2 + 5x} & \leftarrow \text{Subtract } (x)(x + 5) \text{ and bring down } -28. \\ & -2x - 28 & \\ & \underline{-2x - 10} & \leftarrow \text{Subtract } (-2)(x + 5). \\ & -18 & \end{array}$$

Check: $(x + 5)(x - 2) - 18 = x^2 + 3x - 10 - 18$
$= x^2 + 3x - 28$

In this example, the divisor does not divide evenly into the dividend, and the remainder is -18. A polynomial divides evenly when the remainder is 0. The process of division ends when the remainder is 0, or if the degree of the remainder is lower than the degree of the divisor.

Example 2 **Long Division and Factors**

Divide $x^3 - 7x - 6$ by $x + 1$.

Solution

Rewrite the question as a long division. Ensure that the powers are in descending order in both the divisor and the dividend. Include any missing powers by using a coefficient of 0. In this example, there is no x^2-term in the dividend, so add $0x^2$ to the dividend.

$$
\begin{array}{r|l}
 & x^2 - x - 6 \\
x+1 & x^3 + 0x^2 - 7x - 6 \\
 & \underline{x^3 + x^2} \\
 & \quad -x^2 - 7x \\
 & \quad \underline{-x^2 - x} \\
 & \qquad -6x - 6 \\
 & \qquad \underline{-6x - 6} \\
 & \qquad\qquad 0
\end{array}
$$

← Subtract $(x^2)(x + 1)$ and bring down $-7x$.

← Subtract $(-x)(x + 1)$ and bring down -6.

← Subtract $(-6)(x + 1)$.

Check: $(x + 1)(x^2 - x - 6) + 0 = x^3 - x^2 - 6x + x^2 - x - 6$
$= x^3 - 7x - 6$

The last subtraction results in 0. When the remainder is 0, the divisor divides evenly into the dividend. Both the divisor and quotient are factors of the dividend. In this case, $x + 1$ and $x^2 - x - 6$ are factors of $x^3 - 7x - 6$.

Therefore, $x^3 - 7x - 6 = (x + 1)(x^2 - x - 6)$.

EXAMINING THE CONCEPT

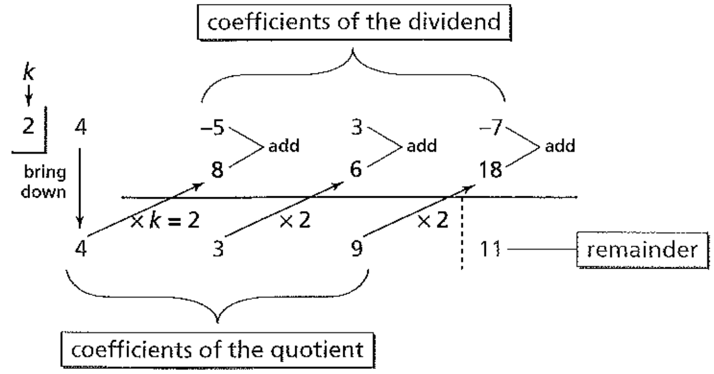
Synthetic Division

Synthetic division is an efficient way to divide a polynomial by a binomial of the form $x - k$, where k is the value that makes the binomial in the divisor equal to 0.

Divide $4x^3 - 5x^2 + 3x - 7$ by $x - 2$. In this case, $k = 2$.

List the coefficients of the dividend, 4, −5, 3, and −7. Bring down the first coefficient of the quotient, which is 4. Multiply by the k-value, which is 2. Add the product to the next coefficient of the dividend: $8 + (-5) = 3$. This result, 3, is the next coefficient of the quotient. Repeat these steps until there are no more coefficients in the dividend.

The process of synthetic division is simpler to use than long division, because it only uses the coefficients of the polynomials involved.



Therefore, $(x - 2)(4x^2 + 3x + 9) + 11 = 4x^3 - 5x^2 + 3x - 7$.

Another way to write this is $\frac{4x^3 - 5x^2 + 3x - 7}{x - 2} = 4x^2 + 3x + 9 + \frac{11}{x - 2}$.

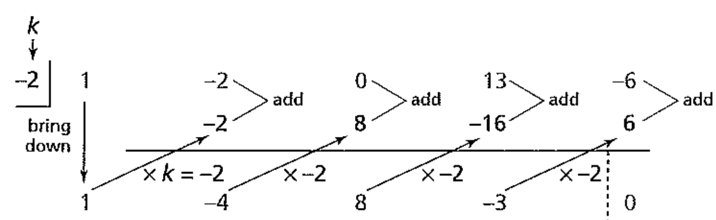
Example 3 **Synthetic Division and Higher Degree Polynomials**

Use synthetic division to divide $13x - 2x^3 + x^4 - 6$ by $x + 2$.

Solution

Rearrange the terms of the dividend in descending order, $x^4 - 2x^3 + 0x^2 + 13x - 6$. Notice that a third term, with a coefficient of 0, has been added to the dividend.

In this case, $k = -2$.



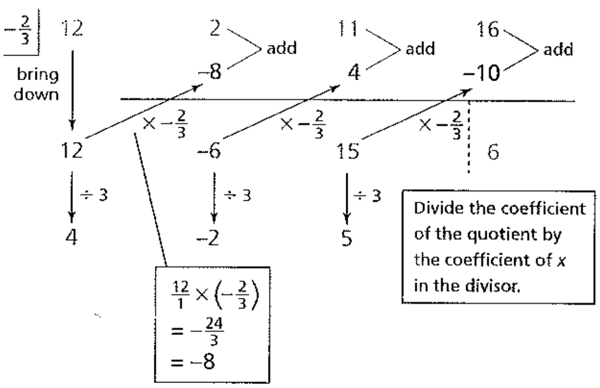
The quotient is $x^3 - 4x^2 + 8x - 3$, and the remainder is 0. Therefore, $(x + 2)(x^3 - 4x^2 + 8x - 3) = x^4 - 2x^3 + 13x - 6$.

You can also use synthetic division when the coefficient of the variable in the divisor is a number other than 1. First determine the number that makes the divisor equal to 0.

Example 4 **More on Synthetic Division**

Use synthetic division to divide $12x^3 + 2x^2 + 11x + 16$ by $3x + 2$.

Solution



The coefficient of x in $3x + 2$ is 3. To find k, let the divisor equal 0 and solve for x.

$$3x + 2 = 0$$
$$3x = -2$$
$$x = -\frac{2}{3}$$

Therefore, $k = -\frac{2}{3}$.

The quotient is $4x^2 - 2x + 5$ and the remainder is 6.

$(3x + 2)(4x^2 - 2x + 5) + 6 = 12x^3 + 2x^2 + 11x + 16$

CHECK, CONSOLIDATE, COMMUNICATE

1. How must the divisor and dividend be arranged for either long division or synthetic division?
2. How can you determine whether the divisor and quotient are factors of the dividend?
3. Write the division statement that shows the relations among the divisor, dividend, quotient, and remainder.

KEY IDEAS

- A polynomial in the form $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x^1 + a_0$ can be divided by another polynomial of degree n or less using long division.
- If the remainder is 0, the divisor and the quotient are factors of the dividend.
- **Synthetic division** is a shortcut for dividing a polynomial in one variable by a binomial. This division yields the same results as long division.

1.4 Exercises

A

1. Rearrange in descending order.
   (a) $2x^5 + 3x - 5x^3 + x^2 - x^4 + 5$ (b) $3x^2 - 2x^4 + 5x - 3 - 2x^3$
   (c) $3x^3 - 5x^4 + 2$ (d) $-3 + 4x^6 + 3x^2 - 2x$
   (e) $-4 + 3x^7 - 4x^2$ (f) $x^4 + 1$

2. Divide using long division.

(a) $x^3 - 3x^2 - 3x + 5$ by $x - 1$ (b) $x^3 - x^2 - 16x - 12$ by $x + 3$

(c) $x^4 - 8x^3 + 2x^2 + 24x + 9$ by $x^2 - 2x - 1$

(d) $x^4 - 10x^2 + 9$ by $x - 1$ (e) $x^4 - 1$ by $x + 1$

(f) $x^5 - x^3 - x^2 + 1$ by $x^2 - 1$

3. Determine the remainder, r, so that the division statement is true.

(a) $(2x - 3)(3x + 5) + r = 6x^2 + x + 5$

(b) $(x + 3)(x + 5) + r = x^2 + 9x - 7$

(c) $(x + 3)(x^2 - 1) + r = x^3 + 3x^2 - x - 3$

(d) $(x^2 + 1)(2x^3 - 1) + r = 2x^5 + 2x^3 + x^2 + 1$

4. Use synthetic division to simplify. State any remainder as a fraction.

(a) $(x^3 - 7x - 6) \div (x - 3)$

(b) $(2x^3 - 7x^2 - 7x + 19) \div (x - 1)$

(c) $(6x^4 + 13x^3 - 34x^2 - 47x + 28) \div (x + 3)$

(d) $(2x^3 + x^2 - 22x + 20) \div (2x - 3)$

(e) $(12x^4 - 56x^3 + 59x^2 + 9x - 18) \div (2x + 1)$

(f) $(6x^3 - 2x - 15x^2 + 5) \div (2x - 5)$

5. **Knowledge and Understanding**

(a) Divide $x^5 + 1$ by $x + 1$ using long division.

(b) Verify your results using synthetic division.

B

6. **Communication:** Create a cubic polynomial division question where the divisor is $x + 3$. Show how each step in synthetic division relates to a step in long division.

7. Create a quartic polynomial question where the divisor is $2x - 3$. Show how each step in synthetic division relates to a step in long division.

8. Determine whether each binomial is a factor of the given polynomial.

(a) $x + 5$, $x^3 + 6x^2 - x - 30$

(b) $x + 2$, $x^4 - 5x^2 + 4$

(c) $x - 2$, $x^4 - 5x^2 + 6$

(d) $2x - 1$, $2x^4 - x^3 - 4x^2 + 2x + 1$

(e) $3x + 5$, $3x^6 + 5x^5 + 9x^2 + 17x - 1$

(f) $5x - 1$, $5x^4 - x^3 + 10x - 10$

9. Use long division to determine the remainder.

(a) $\dfrac{x^3 + 3x^2 - x + 1}{x^2 - 1}$ (b) $\dfrac{4x^4 - 37x^2 - 2}{x^2 - 9}$

(c) $\dfrac{5x^5 - 3x^2 - 2x + 1}{x^3 - x}$ (d) $\dfrac{6x^4 + 31x^3 + 39x^2 - 4x - 2}{x^2 + 5x + 6}$

(e) $\dfrac{x^3 - 3x^2 - x + 3}{x^2 - 2x - 3}$ (f) $\dfrac{x^3 + 8x^2 + 4x - 8}{x^2 + 2x - 3}$

10. When $8x^3 + 4x^2 - px + 6, p \in \mathbf{R}$, is divided by $2x - 1$, the remainder is 3. Determine the value of p.

11. The polynomial $x^3 + px^2 - x - 2, p \in \mathbf{R}$, has $x - 1$ as a factor. What is the value of p?

12. **Application:** The volume of a rectangular box is $(x^3 + 6x^2 + 11x + 6)$ cubic centimetres. The box is $(x + 3)$ cm long and $(x + 2)$ cm wide. How high is the box?

13. A tent has the shape of a triangular prism. The volume of the tent is $(x^3 + 7x^2 + 11x + 5)$ cubic units. The triangular face of the tent is $(2x + 2)$ units wide by $(x + 1)$ units high. How long is the tent?

14. **Check Your Understanding:** In a polynomial division question, the divisor is $2x + 3$, the quotient is $2x^2 + 3x - 4$, and the remainder is -11.
    (a) What is the dividend?
    (b) Verify your answer for (a). Use either long division or synthetic division.

C 15. The volume of a cylindrical can is $(4\pi x^3 + 28\pi x^2 + 65\pi x + 50\pi)$ cm^3. The can is $(x + 2)$ cm high. What is the radius?

16. **Thinking, Inquiry, Problem Solving:** Let $f(x) = x^n - 1$, where n is an integer and $n \geq 1$. Is $f(x)$ always divisible by $x - 1$? Justify your decision.

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

Knowledge and Understanding: Use long division to determine whether each binomial is a factor of $-3x^2 + 2x + x^3 - 24$. Check your answers using synthetic division.

(a) $x - 3$ **(b)** $x + 4$ **(c)** $x - 4$ **(d)** $x - 2$

Application: The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$. A given sphere has a volume of $\left[\frac{4}{3}\pi(x^3 - 3x^2 + 3x - k)\right]$ cubic units and a radius of $(x - 1)$ units. Find the value of k.

Thinking, Inquiry, Problem Solving: The bottom line in this synthetic division gives the coefficients of the quotient. The number at the far right of the bottom line is the remainder.

$$\begin{array}{l|rrrrr} k = -2 & 1 & -3 & 12 & 4 & -11 \\ & & -2 & 10 & -44 & 80 \\ \hline & 1 & -5 & 22 & -40 & \mathbf{69} \end{array}$$

To obtain a remainder of 0, would it be better to try lesser values or greater values of k? Defend your choice.

Communication: Create a table to show the advantages and disadvantages of both synthetic division and long division. Which division method do you prefer, and why?