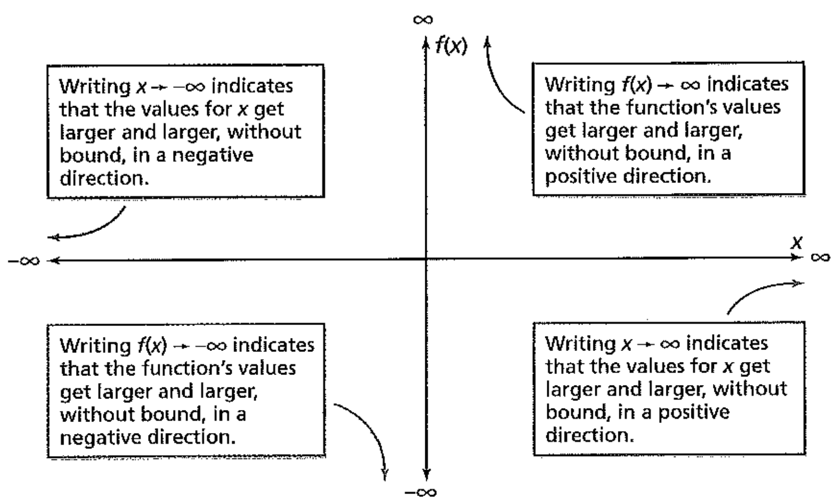
1.2 Investigating the Characteristics of Polynomial Functions

SETTING THE STAGE

Knowing the zeros of a polynomial function can help you sketch the function's graph. You also know that the degree of the function plays a role in defining the shape of the graph. The graph of the linear function $f(x) = 4x - 6$ is a line. The graph of the quadratic function $g(x) = x^2 + 3x - 5$ is a parabola. In most cases, the higher the degree of the polynomial function, the more complex the graph. In this section, you will examine the graphs of more complex polynomial functions.

EXAMINING THE CONCEPT

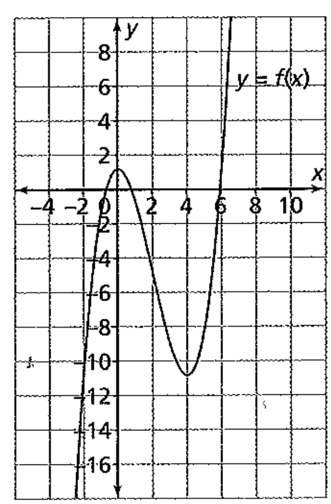
Characteristics of Polynomial Functions



An important characteristic of any polynomial function is its **end behaviour**. End behaviour describes the values of a function, $f(x)$, as x takes on large positive, or large negative, numbers. You can describe end behaviour using the symbols ∞ and $-\infty$, which mean positive and negative infinity, respectively. The diagram on the left shows how these symbols can be used.

Example 1 **Describing the End Behaviour of a Polynomial Function from Its Graph**

Describe the end behaviour of $f(x)$ as $x \to -\infty$ and $x \to \infty$.



Solution

As $x \to \infty$, the values of $f(x)$ get larger in the positive direction. In symbols, $f(x) \to \infty$.

As $x \to -\infty$, the values of $f(x)$ get larger in the negative direction. In symbols, $f(x) \to -\infty$.

Investigating the End Behaviour and Turning Points

1. Using graphing technology, adjust the window settings so that the intervals $-5 \le x \le 5$ and $-40 \le y \le 20$ are on the axes.

2. Graph each function, and then copy and complete the table.
   (a) $f(x) = 9x^2 - 8x - 2$
   (b) $f(x) = -x^4 - 3x^3 + 3x^2 + 8x + 5$
   (c) $f(x) = 2x^6 - 13x^4 + 15x^2 + x - 17$
   (d) $f(x) = -2x^4 - 4x^3 + 3x^2 + 6x + 9$
   (e) $f(x) = x^3 - 5x^2 + 3x + 4$
   (f) $f(x) = 2x^5 + 7x^4 - 3x^3 - 18x^2 - 20$
   (g) $f(x) = -x^7 + 8x^5 - 16x^3 + 8x$
   (h) $f(x) = -2x^3 + 8x^2 - 5x + 3$

Function	Degree	Number of Turning Points	Leading Coefficient: Positive or Negative?	Degree: Even or Odd?	End Behaviour as $x \to -\infty$	End Behaviour as $x \to \infty$
(a)						
(b)						
...	...	...	...	...	...	...
(h)						

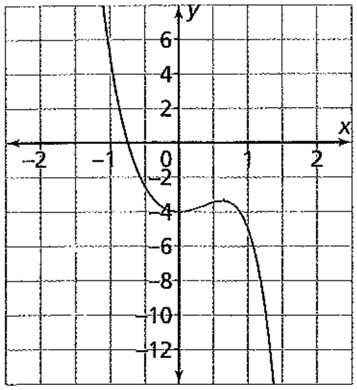
3. Make a conjecture about the maximum number of turning points in the graph of a polynomial function with degree 8, 9, or n.

4. Make a conjecture about the end behaviour of a function with a degree that is **(a)** even and **(b)** odd.

5. Make a conjecture about the end behaviour of a function with a degree that is
   (a) even and has a positive leading coefficient
   (b) even and has a negative leading coefficient
   (c) odd and has a positive leading coefficient
   (d) odd and has a negative leading coefficient

Example 2 **Describing the End Behaviour of a Polynomial Function from Its Equation**

Describe the end behaviour of $f(x) = -2x^5 - 3x^3 + 4x^2 - 4$ as $x \to -\infty$ and $x \to \infty$.



$f(x) = -2x^5 - 3x^3 + 4x^2 - 4$

Solution

The degree of the function is an odd number, 5. Therefore, the end behaviours are opposite. The leading coefficient is a negative number, -2, and as $x \to \infty$, the values of $f(x)$ get larger in the negative direction. As $x \to -\infty$, the values of $f(x)$ get larger in the positive direction. In other words,

$$\text{as } x \to \infty, f(x) \to -\infty, \text{ and}$$
$$\text{as } x \to -\infty, f(x) \to \infty$$

The graph of the function verifies the analysis.

Investigating the Number of Zeros

1. Using graphing technology, adjust the window settings so that the intervals $-10 \le x \le 10$ and $-10 \le y \le 10$ are on the axes.
2. State the degree of each polynomial and then graph it. Determine the number of zeros.
   (a) $f(x) = x^3 - 2x^2 - 4x + 8$ (b) $f(x) = x^3 + x^2 - 2x - 7$
   (c) $f(x) = x^3 + 2x^2 - 3x - 5$
3. Before graphing, clear all previous functions. State the degree of each polynomial and then graph it. Determine the number of zeros.
   (a) $f(x) = x^4 + 2x^3 - x^2 - 2x$ (b) $f(x) = -x^4 + 2x^3 + x^2 + 2x$
   (c) $f(x) = 2x^4 - 6x^3 + x^2 + 4x + 5$ (d) $f(x) = -x^4 - x^3 + 3x^2 + x - 2$
   (e) $f(x) = x^4 + x^3 + x + 1$
4. Make a conjecture about the minimum and maximum number of zeros possible for a polynomial function with degree
   (a) 5 (b) 6 (c) 7 (d) 8 (e) n

Investigating Finite Differences in Polynomial Functions

Recall that for a linear function, the rate of change is constant, so the first differences are constant. The rate of change for a quadratic function is variable, but the second differences are constant.

Does the degree of a polynomial function influence the value of the finite differences?

1. Make a conjecture about the finite differences of a cubic polynomial.
2. To verify your conjecture, create a table with the first, second, and third differences for $f(x) = 2x^3 - 3x^2 + 4x - 1$, where $-2 \le x \le 5$.

Use either pencil and paper or a graphing calculator. If you are using a TI-83 Plus, follow these steps:

(a) Press [STAT] [1] and enter the x-values into **L1**.

(b) Scroll right and up to select **L2**. Enter the function, using **L1** as the variable x. Press [ALPHA] [+] [2] [2nd] [1] [^] [3] [−] [3] [2nd] [1] [^] [2] [+] [4] [2nd] [1] [−] [1] [ALPHA] [+]

(c) Press [ENTER] to display the values of the function in **L2**.

(d) Find the first differences. Scroll right and up to select **L3**. Then press [ALPHA] [+] [2nd] [STAT]. Scroll right to **OPS** and press [7] to choose **ΔList(**. Enter **L2** by pressing [2nd] [2] [)] [ALPHA] [+]. Press [ENTER] to see the first differences displayed in **L3**.

(e) Repeat step (d) to find the second differences. In **L4**, apply **ΔList(** to **L3**. To find the third differences in **L5**, apply ΔList(to **L4**. **L1** contains the values of x, **L2** the values of $f(x)$, **L3** the first differences, **L4** the second differences, and **L5** the third differences.

3. Choose another cubic polynomial function and create the differences table. If you are using the TI-83 Plus, simply scroll right and up to highlight **L2** and then press [ENTER]. Then edit the original function. Enter the new coefficients and press [ENTER] to view the new table.

4. Choose a polynomial function of degree 4 and create a differences table.

5. Choose a polynomial function of degree 5 and create a differences table.

6. What conclusion can you make about the finite differences of a polynomial function with degree n?

7. What relation exists between the leading coefficient of a polynomial of degree n and its common difference?

CHECK, CONSOLIDATE, COMMUNICATE

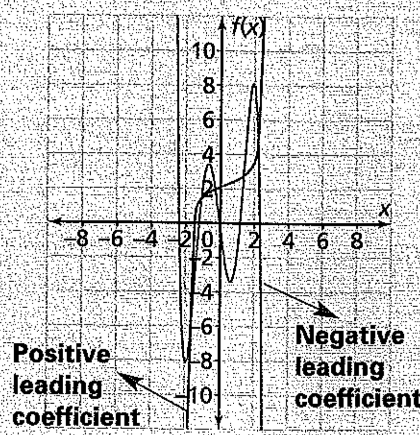
1. What is the maximum number of turning points in the graph of a polynomial of degree 1? degree 2? degree 3? degree 20? degree n?
2. Describe the end behaviour of an even-degree polynomial with a positive leading coefficient and with a negative leading coefficient.
3. Describe the end behaviour of an odd-degree polynomial with a positive leading coefficient and with a negative leading coefficient.
4. Explain why there is at least one x-intercept in the graph of an odd-degree polynomial.
5. A degree-4 polynomial has a leading coefficient of -2. What is the value of the fourth finite difference?

KEY IDEAS

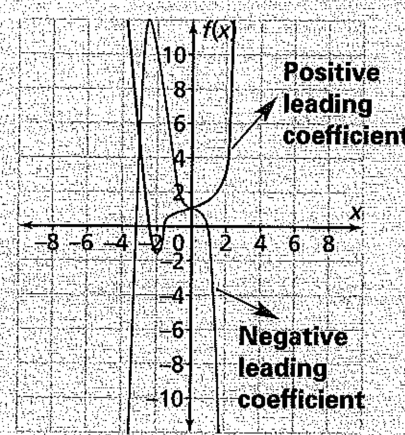
- Polynomial functions behave differently, depending on the degree. If the degree is even, the end behaviours are the same. If the degree is odd, the end behaviours are opposite.

- The degree and the leading coefficient determine the end behaviours of a polynomial function.

Odd Degree



Even Degree



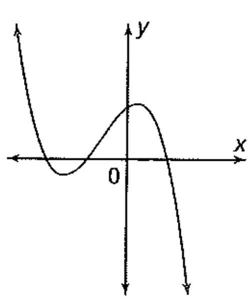
- The graphs of polynomial functions of degree n may have
  0 to n x-intercepts when n is even.
  1 to n x-intercepts when n is odd.
- The graphs of polynomial functions of degree 4 can be W-shaped curves.
- The graphs of polynomial functions of degree n have at most $n - 1$ turning points.
- The nth finite differences of polynomial functions of degree n are constant.
- The third finite difference of a degree-3 polynomial is related to the leading coefficient by a factor of $3 \times 2 \times 1$.
- The fourth finite difference of a degree-4 polynomial is related to the leading coefficient by a factor of $4 \times 3 \times 2 \times 1$.

1.2 Exercises

A

1. Use the graph of each polynomial function to identify the polynomial as cubic or quartic, state the sign of the leading coefficient of its function, describe the end behaviour, and say whether the graph has a turning point.

(a)



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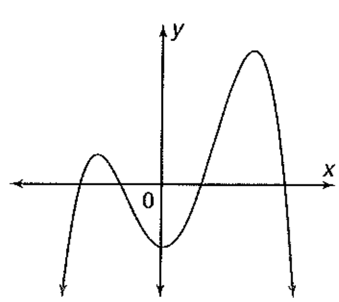
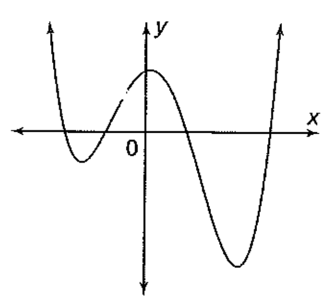
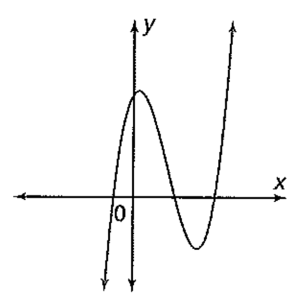


(c)



(d)



12. Sketch the graph of a polynomial function that satisfies each set of conditions.

(a) degree 4, positive leading coefficient, 3 zeros, 3 turning points

(b) degree 4, negative leading coefficient, 2 zeros, 1 turning point

(c) degree 4, positive leading coefficient, 1 zero, 3 turning points

(d) degree 3, negative leading coefficient, 1 zero, no turning point

(e) degree 3, positive leading coefficient, 2 zeros, 2 turning points

13. Let $f(x) = 2(x - 3)^2 (x + 1)$.

(a) Sketch a possible graph.

(b) Confirm your sketch using a graphing calculator.

(c) Sketch another, similar type of graph that uses the other zero as a turning point. How will the original equation change?

14. State the third finite difference for each polynomial. Verify the value with a difference table.

(a) $f(x) = -2x^3 + 4x^2 - 3x - 2$ (b) $f(x) = 3(x - 1)(x + 2)(x - 6)$

15. State the fourth finite difference for each polynomial. Verify the value with a difference table.

(a) $f(x) = 0.25x^4 - 0.15x^3 + 1.5x^2 - 0.5x + 3$

(b) $f(x) = -0.15(x + 2)(x + 3)(2x - 1)(3x - 2)$

16. **Check Your Understanding:** Copy and complete the table.

Degree of $f(x)$	Sign of Leading Coefficient of $f(x)$	End Behaviour of $f(x)$ as $x \to \infty$	End Behaviour of $f(x)$ as $x \to -\infty$
odd	positive		
even	negative		
odd	negative		
even	positive		

17. **Thinking, Inquiry, Problem Solving:** You can use finite differences to determine the leading coefficient of any polynomial function. Determine the relation that exists between the nth finite difference and the leading coefficient, a, for the general polynomial function $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$.

18. The polynomial $f(x) = ax^5 + 3x^4 - 2x^3 - 3x^2 + x - 1$ has a common difference of -120. What is the value of a?

19. Write a polynomial function of degree 5 such that $f(x) \to \infty$ as $x \to -\infty$. Graph $f(x)$ using graphing technology. Determine the coordinates of all turning points. Identify the intervals in which $f(x)$ is increasing and decreasing.

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

Knowledge and Understanding: Let $f(x) = x^4 - 5x^2 + 4$.

(a) Determine the end behaviour as $x \to \infty$ and $x \to -\infty$, the maximum number of zeros, and the maximum number of turning points.

(b) Use graphing technology to graph f and confirm your answers for (a).

Application: The Adanac Clothing Company makes scarves and sells them wholesale. A distributor pays $15 per scarf to buy 10 scarves. The price decreases by $1 per scarf if the distributor buys 5 more scarves. The price continues to drop by $1 per scarf for every additional 5 scarves bought. Determine a polynomial function to represent the revenue. Use the function to find the number of scarves that Adanac should sell to maximize revenue.

Number of Scarves Sold	Revenue ($)
10	150
15	210
20	260
25	300
⋮	⋮

Thinking, Inquiry, Problem Solving

The function $g(x) = x^4 - 2x^3 - 5x^2 - 4x + 4$ has two zeros. The intervals in which the zeros lie can be determined without graphing the function. Identify the intervals with the zeros from the choices below. Defend your choices using mathematical reasoning.

(a) $-1 \leq x \leq 0$ **(b)** $0 \leq x \leq 1$ **(c)** $1 \leq x \leq 2$

(d) $2 \leq x \leq 3$ **(e)** $3 \leq x \leq 4$ **(f)** $4 \leq x \leq 5$

Communication: A classmate was absent when this section was covered. Describe to her or him at least five pieces of information you can obtain about the graph of a polynomial function by examining the function. Be sure to describe where you are looking, what you are doing, and why you are doing it to find the information. Use $h(x) = 2x - 4 - 5x^4 + 3x^3$ to guide your description.

The Chapter Problem

Developing a Model for Canada's Population

CP5. Using graphing technology, create a scatter plot. Enter the years since 1851 into **L1** and the population into **L2**. Use linear regression and quadratic regression to find two different polynomial models.

CP6. What is the end behaviour of each model? Which model has a turning point?

CP7. Use each model to determine the population in 2001. Which model is the better predictor based on the actual population of Canada in 2001?

CP8. Use each model to determine the population in 2006 and 2016. Which model might be the better predictor? Explain.