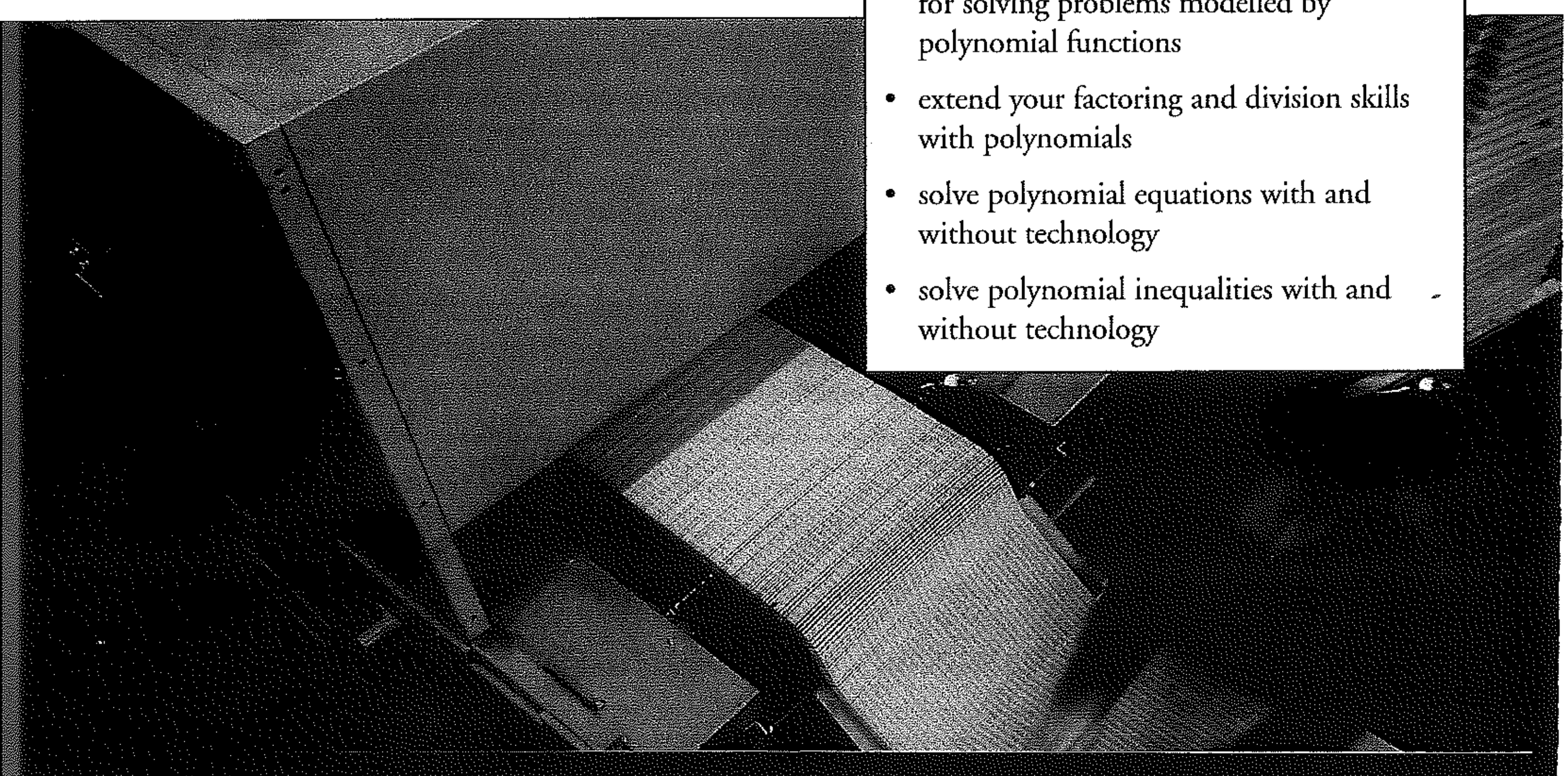
1 Polynomial Function Models

One goal of mathematics is to find functions to model data. If the model is a good fit, you can use it to analyze present behaviour as well as to predict past and future behaviour. In previous courses, you studied linear and quadratic functions. Both these functions are part of a much larger group of functions called polynomials. This chapter will extend your earlier studies beyond degree-2 functions.

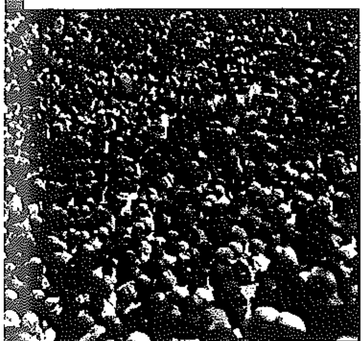
In this chapter, you will

- investigate polynomial functions and study their properties
- examine the nature of change in polynomial functions
- sketch the graphs of polynomial functions
- model data using graphs of polynomial functions
- extend the notion of polynomial functions beyond linear and quadratic functions
- develop the algebra and technology skills for solving problems modelled by polynomial functions
- extend your factoring and division skills with polynomials
- solve polynomial equations with and without technology
- solve polynomial inequalities with and without technology



The Chapter Problem

Developing a Model for Canada's Population



The population of Canada is measured on a regular basis through the taking of a census. The table shows the population of Canada at the end of each period. From 1851 to 1951, each period is a ten-year interval. From 1951 to 2001, each period is a five-year interval.

Period	Census Population at the End of a Period (in thousands)
1851–1861	3 230
1861–1871	3 689
1871–1881	4 325
1881–1891	4 833
1891–1901	5 371
1901–1911	7 207
1911–1921	8 788
1921–1931	10 377
1931–1941	11 507
1941–1951	13 648

Period	Census Population at the End of a Period (in thousands)
1951–1956	16 081
1956–1961	18 238
1961–1966	20 015
1966–1971	21 568
1971–1976	23 450
1976–1981	24 820
1981–1986	26 101
1986–1991	28 031
1991–1996	29 672
1996–2001	30 755

Source: Statistics Canada, Demography Division

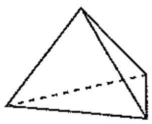
How can you use this data to predict the population of Canada over the next five years? To answer this question, you could organize the data using models called **polynomials**. There are several types of polynomial functions. You can use any polynomial to make predictions, but each type has its own strengths and weaknesses.

You will investigate these questions throughout this chapter: What are the various polynomial models, and how can you use them to predict the population of Canada over the next five to fifty years? Which model is the most accurate predictor? What things could happen that could alter the model? What are the limitations of a polynomial model?

For help with this problem, see pages 18, 27, 38, 53, 64, and 74.

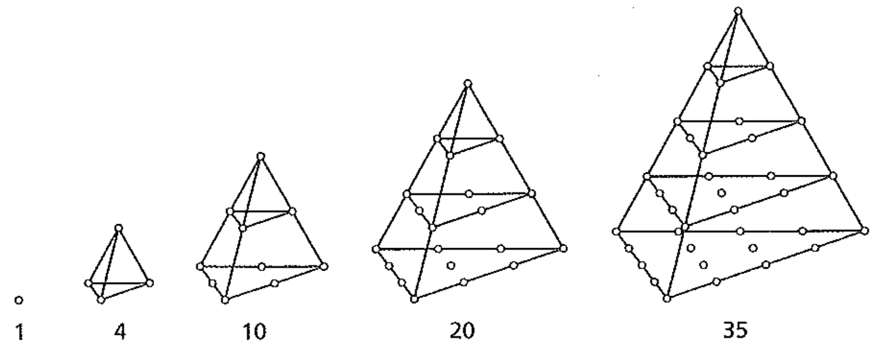
Chapter Challenges

Challenge 1



tetrahedron

The numbers 1, 4, 10, 20, and 35 are called tetrahedral numbers because they are related to a four-sided shape called a tetrahedron.



(a) Determine the next three tetrahedral numbers.

(b) You could use a polynomial to generate more tetrahedral numbers. Use finite differences to determine the degree of this polynomial.

(c) Determine the polynomial model that you can use to generate tetrahedral numbers.

(d) What is the 50th tetrahedral number?

(e) Is 47 850 a tetrahedral number? Justify your answer.

Challenge 2

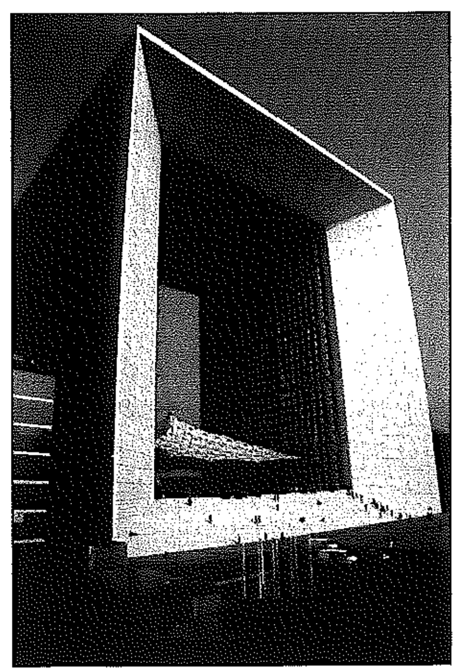
Let $S(n)$ be the sum of the cubes of the integers from 1 to n such that

$$S(n) = 1^3 + 2^3 + 3^3 + \ldots + n^3$$

(a) Determine the polynomial model that you can use to calculate the sum of the cubes of the first n integers.

(b) Determine $S(1000)$.

(c) Prove that $S(n)$ is always a perfect square.



GETTING READY

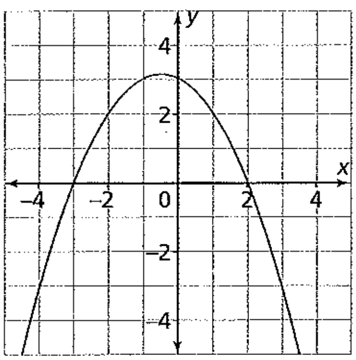
In this chapter, you will be working with algebraic expressions and functions that require many of the same skills used for linear and quadratic functions. These exercises will help you warm up for the work ahead.

1. Evaluate for $x = -2$.
   (a) $f(x) = -3x + 5$
   (b) $f(x) = (4x - 2)(3x - 6)$
   (c) $f(x) = -3x^2 + 2x - 1$
   (d) $f(x) = (5x + 2)^2$
2. State the conjugate of each number.
   (a) $3 + 2i$
   (b) $-1 - 3i$
   (c) 4
   (d) $16i$
3. Expand, simplify, and express in standard form.
   (a) $(x - 6)(x + 2)$
   (b) $(5 - 2x)(3 + 4x)$
   (c) $x(5x - 3) + 2x(3x + 2)$
   (d) $(x - 1)(x + 3) - (2x + 5)(x - 2)$
4. Factor.
   (a) $x^2 - 8x + 12$
   (b) $2x^2 + 5x - 3$
   (c) $6x^2 + 17x + 5$
   (d) $6x^2 + 11x^2 + 4$
   (e) $4x^2 - 25$
   (f) $2x(x + 1) - 5(x + 1)$
5. Solve.
   (a) $(x + 4)(x - 2) = 0$
   (b) $(2x - 5)(3x + 8) = 0$
   (c) $x^2 - 5x - 6 = 0$
   (d) $6x^2 + 7x - 5 = 0$
   (e) $3x^2 = 13x + 10$
   (f) $10x^2 - 4x - 8 = 2x^2 - 2x + 7$
6. Solve using the quadratic formula. Round your answers to two decimal places.
   (a) $9x^2 - x - 7 = 0$
   (b) $41x^2 + x = 31$
   (c) $-18x^2 + 9x = -23$
7. Expand and simplify.
   (a) $(x - i)(x + i)$
   (b) $(2x - 3i)(2x + 3i)$
   (c) $[x - (2 + i)][x - (2 - i)]$
   (d) $(2i - 5)^2$
8. Determine the roots of each equation.
   (a) $x^2 + 4 = 0$
   (b) $x^2 - 2x + 13 = 0$
   (c) $x^2 - 6x + 14 = 0$
9. Solve and graph the solution set for each inequality.
   (a) $6 - 2x > x - 6$
   (b) $1 \leq x + 3 < 5$
   (c) $x^2 \leq 36$

10. Let $f(x) = 3x - 2$.
   (a) Graph $f(x)$.
   (b) Describe $f(x)$ when $x > 3$.
   (c) State the restriction on x when $4 \leq f(x) \leq 10$.

11. Graph $f(x) = -2x^2 - 4x + 16$ by hand. Explain the properties of quadratic functions you used to draw the graph.

12. A T-ball player hits a baseball from a tee that is 0.5 m tall. The flight of the ball is modelled by $h(t) = -4.9t^2 + 6t + 0.5$, where h is the height in metres at t seconds.
   (a) Once the ball is hit, how long is it in the air?
   (b) When does the ball reach its maximum height?
   (c) What is the maximum height?



13. Determine the equation of the function from its graph on the left.

14. The graph of a quadratic function passes through points $(-4, 0)$ and $(3, 0)$.
   (a) Determine an equation that represents all such curves.
   (b) Determine the equation of the graph that also passes through $(5, -36)$.

15. The population of Canada from 1971 to 1996 is shown.

Period	1971–1976	1976–1981	1981–1986	1986–1991	1991–1996
Population × 1000	23 450	24 820	26 101	28 031	29 672

Source: Statistics Canada

   (a) Use a graphing calculator to draw a scatter plot of the data. What pattern does the scatter plot appear to follow?
   (b) Determine a quadratic equation that models the data.
   (c) Graph the equation.
   (d) Predict the population for 1998.

16. A model rocket is shot straight up from the roof of a building 21 m high. The table shows the height of the rocket during the first two seconds.

Time (s)	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Height (m)	21.00	24.44	27.28	29.49	31.10	32.09	32.48	32.24	31.40

   (a) Use a graphing calculator to draw a scatter plot of the data.
   (b) Determine a quadratic equation that models the data.
   (c) Graph the equation.
   (d) When will the rocket hit the ground?

1.1 Polynomial Functions

SETTING THE STAGE

In earlier math courses, you used linear functions, quadratic functions, and trigonometric functions to model real-world problems. Are there any other types of functions that you can use to model real-world relations?

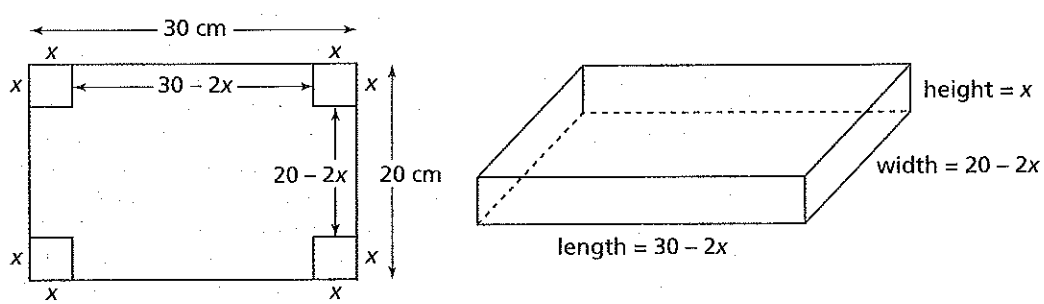
A toy manufacturer has created a new card game. Each game is packaged in an open-top cardboard box, which is then wrapped with clear plastic. The box for the game is made from a 20-cm by 30-cm piece of cardboard. Four equal squares are cut from the corners, one from each corner, of the cardboard piece. Then the sides are folded and the edges that touch are glued. What must be the dimensions of each square so that the resulting box has maximum volume?

You will need a mathematical model to represent the situation. In this section, you will examine a new class of functions called **polynomials**.

EXAMINING THE CONCEPT

Defining a Polynomial

To develop a model for this problem, begin by defining the variables. Let x represent the side length of each square cut from the cardboard. Let V represent the volume of the box. Then draw and label diagrams.



Two-dimensional and three-dimensional sketches of the cardboard box

Volume	$= \text{length} \times \text{width} \times \text{height}$	
$V(x)$	$= (30 - 2x)(20 - 2x)(x)$	Expand by multiplying the binomials.
	$= (600 - 60x - 40x + 4x^2)(x)$	Expand by multiplying by the monomial.
	$= 600x - 60x^2 - 40x^2 + 4x^3$	Collect like terms.
	$= 600x - 100x^2 + 4x^3$	Rearrange powers of x in descending order.
	$= 4x^3 - 100x^2 + 600x$	

The resulting algebraic model $V(x) = 4x^3 - 100x^2 + 600x$ is an example of an equation that contains a **polynomial**.

Definition of a Polynomial in One Variable

A polynomial is an expression of the form
$a_nx^n + a_{n-1}x^{n-1} + \dots + a_2x^2 + a_1x + a_0$, where $a_0, a_1, \dots, a_n$ are real numbers and n is a natural number. The coefficient of the term of highest degree in a polynomial is called the leading coefficient.

Any polynomial that is written with descending powers of the variable is in **standard form**.

Degree of a Polynomial

The degree of a polynomial is the value of the highest exponent of the variable.

A **linear function** has the form $f(x) = mx + b$. The functions $f(x) = 3x - 5$, $g(x) = \left(-\frac{2}{3}\right)x + 4$, and $h(x) = 9x$ are all linear. In each of these functions, the right side is a polynomial of degree 1.

Linear functions belong to the larger class of functions called **polynomial functions**.

Polynomial Function

A polynomial function is a function whose equation is defined by a polynomial in one variable.

Polynomials of degree 5 or greater do not have special names. They are simply called polynomials.

A **quadratic function** in the form $f(x) = ax^2 + bx + c$, $a \neq 0$, is a polynomial function of degree 2.

The model for the volume of the cardboard box, $V(x) = 4x^3 - 100x^2 + 600x$, is a degree-3 polynomial function, called a **cubic function**. This function is in the form $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$. The polynomial function $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, $a \neq 0$, is a **quartic function** and has degree 4.

Example 1 **Standard Form and the Degree of a Polynomial Function**

Express $f(x) = 2x + 5 - 3x^3 - 9x^2$ in standard form and state its degree.

Solution

Rearrange the terms in descending order: $f(x) = -3x^3 - 9x^2 + 2x + 5$.
The largest exponent in $f(x)$ is 3. The function has degree 3, so it is a cubic function.

You can use polynomial functions to model many real-life situations. For example, the function $h(t) = -4.9t^2 + 20t + 1.6$ models the height, h, in metres of a ball above the ground t seconds after the ball has been thrown into the air.

Example 2 **Evaluating Polynomial Functions**

A 2-cm by 2-cm square is cut from each corner of a 20-cm by 30-cm piece of cardboard. What is the volume of the corresponding cardboard box? The polynomial function $V(x) = 4x^3 - 100x^2 + 600x$ models the volume, V, in cubic centimetres and x represents the side length of each square that is cut from each corner.

Solution

To determine the volume, substitute $x = 2$ and evaluate.

$$\begin{aligned} V(x) &= 4x^3 - 100x^2 + 600x \\ V(2) &= 4(2)^3 - 100(2)^2 + 600(2) \\ &= 4(8) - 100(4) + 1200 \\ &= 832 \end{aligned}$$

The volume of the box will be 832 cm^3 if a 2-cm by 2-cm square is cut from each corner of the piece of cardboard.

EXAMINING THE CONCEPT

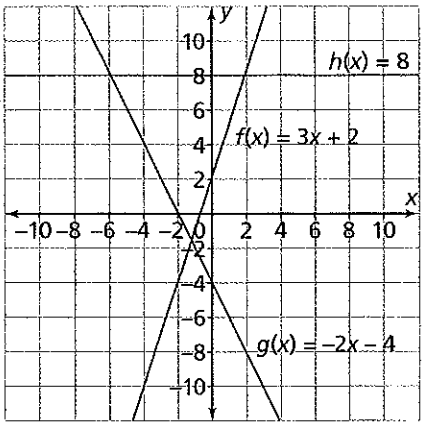
Describing Polynomial Functions

You can describe any function in terms of the values of $f(x)$ for a specific interval of the domain. The symbol $f(x)$ in this case means the value of the function at a number x. The symbol $f(x)$ is also the name of the function, which may be called just f. The function is **increasing** if the graph of the function *rises* from left to right along the x-axis. The function is **decreasing** if the graph of the function *falls* from left to right along the x-axis.

Increasing and Decreasing Functions

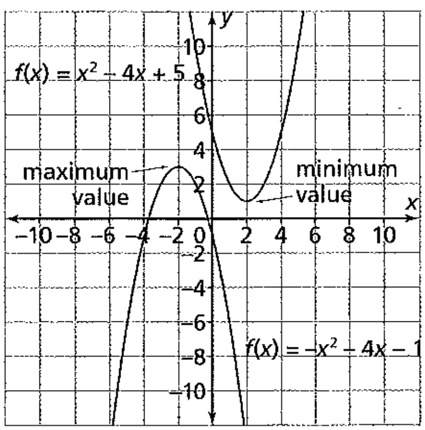
A function f is **increasing** on an interval I if $f(x_1) < f(x_2)$, $x_1 < x_2$, and x_1 and x_2 are in I. A function f is **decreasing** on an interval I if $f(x_1) > f(x_2)$, $x_1 < x_2$, and x_1 and x_2 are in I.

Intuitively, a function is increasing if the graph rises from left to right and decreasing if the graph falls from left to right.



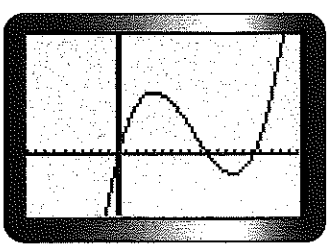
Linear functions with degrees 0 and 1.

The graph of the linear function $f(x) = 3x + 2$ has a positive slope, so the function *increases* over the domain of the set of real numbers. The graph of $g(x) = -2x - 4$ has a negative slope, so the function *decreases* over the same domain. The horizontal line that represents $h(x) = 8$ has slope 0, so the function is neither increasing nor decreasing.



Quadratic functions with degree 2.

The top parabola opens up, and the bottom parabola opens down. The quadratic function of the parabola that opens up decreases to a minimum value (at the vertex) and then increases. The quadratic function of the parabola that opens down increases to a maximum value (at the vertex) and then decreases.



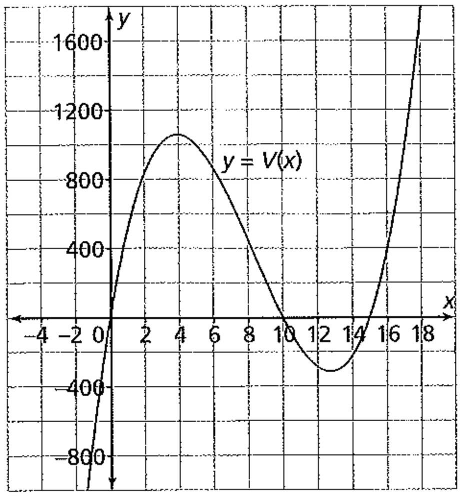
$V(x) = 4x^3 - 100x^2 + 600x$

To quickly graph a cubic function use graphing technology. Enter the model for the volume of the cardboard box, $V(x) = 4x^3 - 100x^2 + 600x$, into the equation editor of a graphing calculator. Set the window to **Xmin=–10, Xmax=20, Ymin=–1000, Ymax=2000** to produce this graph.

The graph of $V(x)$ is a continuous curve that rises, falls, and then rises again. $V(x)$ is increasing on the intervals $x < 4$ and $x > 13$. $V(x)$ is decreasing on the interval $4 < x < 13$. As a result, the graph turns at two different points.

Turning Point

A turning point is a point on a curve that is higher or lower than all nearby points. A turning point occurs where a function changes from increasing to decreasing or vice versa. When the function $y = f(x)$ changes from increasing to decreasing at (x, y), then (x, y) is called a **local maximum** and $f(x) = y$ is the **local maximum value**. When the function $y = f(x)$ changes from decreasing to increasing at (x, y), then (x, y) is called a **local minimum** and $f(x) = y$ is the **local minimum value**.



$V(x)$ has a local maximum and a local minimum.

The graph of $V(x)$ has two turning points at about $(4, 1056)$ and $(13, -315)$. The local maximum is $(4, 1056)$ and the local minimum is $(13, -315)$. The maximum and minimum values are 1056 and -315, respectively. The graph also has three x-intercepts, $x = 0$, $x = 10$, and $x = 15$. These values correspond to the real zeros of the function.

The zeros of a function occur when $f(x) = 0$.
To find the zeros, factor $V(x)$ and set each factor to 0.

$$V(x) = (30 - 2x)(20 - 2x)(x)$$

$$\begin{array}{lll} 30 - 2x = 0 \quad \text{or} & 20 - 2x = 0 \quad \text{or} & x = 0 \\ 30 = 2x & 20 = 2x & \\ 15 = x & 10 = x & \end{array}$$

The terms zeros and x-intercepts can be used interchangably.

The zeros define intervals where $V(x)$ is positive or negative. The graph lies *below* the x-axis for $x < 0$ and $10 < x < 15$, and $V(x)$, or the volume of the box, is negative. The graph lies *above* the x-axis for $0 < x < 10$ and $x > 15$, and $V(x)$, or the volume of the box, is positive.

The values of $V(x)$ must be positive because the model describes volume, which can have only positive values. If $x > 10$, a box cannot be created at all! The only realistic interval is $0 \le x \le 10$.

Then the appropriate model for the volume of the cardboard box is $V(x) = (30 - 2x)(20 - 2x)(x)$, where $0 < x < 10$. The model is a polynomial function with a **restricted domain**. In this case, the unrestricted domain of $V(x)$ is the set of real numbers.

Simplify $V(x)$ further by factoring.

$$\begin{aligned} V(x) &= (30 - 2x)(20 - 2x)(x) \\ &= -2(-15 + x)(-2)(-10 + x)(x) \quad \text{Simplify.} \\ &= 4x(x - 15)(x - 10) \end{aligned}$$

In factored form, the zeros of the function are clearly visible.

Zeros of Polynomial Functions in Factored Form

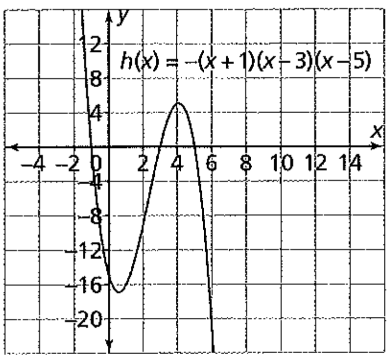
The zero of a **linear** function $f(x)$ is s in $f(x) = k(x - s)$.
The zeros of a **quadratic** function $f(x)$ are s and t in $f(x) = k(x - s)(x - t)$.
The zeros of a **cubic** function $f(x)$ are s, t, and u in $f(x) = k(x - s)(x - t)(x - u)$.

Example 3 **Sketching the Graphs of Polynomial Functions in Factored Form**

For the polynomial function $h(x) = -(x + 1)(x - 3)(x - 5)$,

i. determine the zeros **ii.** sketch the graph **iii.** describe the shape

Solution



$h(x) = -(x + 1)(x - 3)(x - 5)$ is in factored form. The function is cubic and has zeros -1, 3, and 5. The turning points do not necessarily occur at the midpoint between each pair of x-intercepts, as they do for a quadratic function. To complete the graph, determine more points. Create a table with x-values to the left of the smallest zero, between the zeros, and to the right of the largest zero. The more points used, the better the sketch.

x	−3	−2	0	1	2	4	6	7
$h(x)$	96	35	−15	−16	−9	5	−21	−64

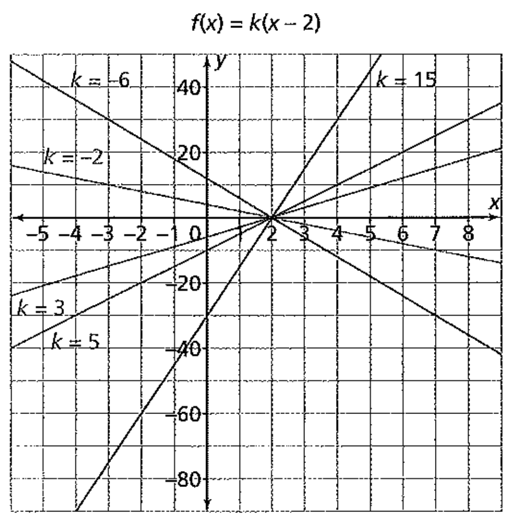
This function decreases when $x < 0.5$ and $x > 4$. It increases when $0.5 < x < 4$. The local minimum is $(0.5, -17)$ and the local maximum is $(4, 5)$.

EXAMINING THE CONCEPT

Families of Polynomial Functions and the Role of *k*

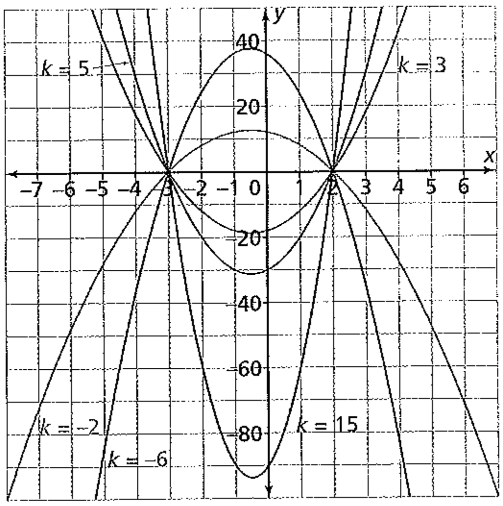
Any linear function $f(x) = k(x - s)$ has a zero at s and slope k.
When $k > 0$, f increases and when $k < 0$, f decreases.
When $k = 0$, the graph of the function (a line) is horizontal.

A family of linear functions is created if s is constant and k varies. The graphs of the family of functions pass through the same x-intercept.



A family of linear functions

$f(x) = k(x - 2)(x + 3)$

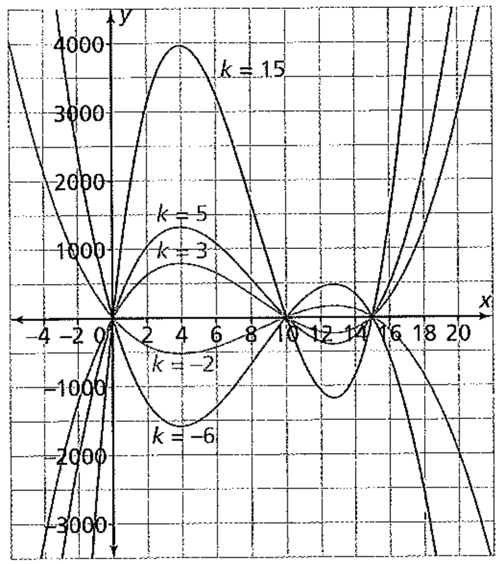


A family of quadratic functions

Any quadratic function $f(x) = k(x - s)(x - t)$ has zeros s and t. When $k > 0$, the parabola opens up. The function decreases and then increases from left to right. When $k < 0$, the parabola opens down. The function increases and then decreases from left to right. A family of quadratic functions is created if s and t are constant and k varies. The graphs of the family of functions pass through the same x-intercepts.

The volume function $V(x) = 4x(x - 15)(x - 10)$ is in factored form. Look at similar cubic functions with the general factor k, where $k \in \mathbf{R}$, that is, $f(x) = kx(x - 15)(x - 10)$.

$f(x) = kx(x - 15)(x - 10)$

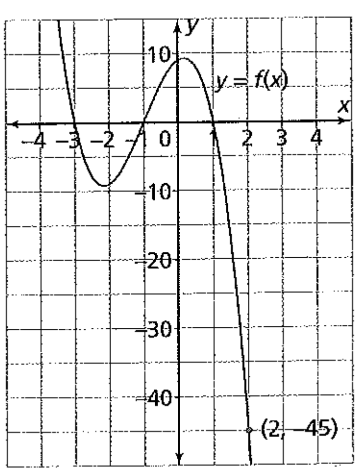


A family of cubic functions

The cubic function $f(x) = k(x - s)(x - t)(x - u)$ has zeros s, t, and u. When $k > 0$, the function increases, decreases, and increases again, from left to right. When $k < 0$, the function decreases, increases, and decreases again, from left to right.

Since a family of polynomial functions is possible for a set of given zeros, knowing the coordinates of an additional point on the graph allows you to determine the equation of a specific function.

Example 4 Determining the Equation of a Polynomial Function from Its Graph



Determine the equation of $f(x)$. Express the equation in standard form.

Solution

The function has three zeros, −3, −1, and 1, and the corresponding factors are $x + 3$, $x + 1$, and $x - 1$. Because the function decreases, increases, and decreases again, the function is cubic and the value of k is negative. The function is of the form $f(x) = k(x + 3)(x + 1)(x - 1)$. Substitute the coordinates of a point on the curve, for example, (2, −45), and solve for k.

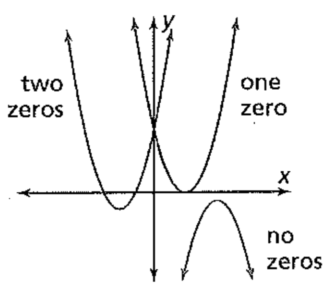
$$-45 = k(2 + 3)(2 + 1)(2 - 1) \quad \text{Solve for } k.$$
$$-45 = 15k$$
$$-3 = k$$

$$f(x) = -3(x + 3)(x + 1)(x - 1)$$
$$= -3(x + 3)(x^2 - 1) \quad \text{Expand.}$$
$$= -3(x^3 - x + 3x^2 - 3)$$
$$= -3x^3 - 9x^2 + 3x + 9$$

The equation of the function in standard form is $f(x) = -3x^3 - 9x^2 + 3x + 9$.

EXAMINING THE CONCEPT

A Closer Look at Zeros of Polynomial Functions



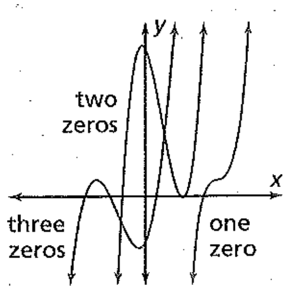
A quadratic function can have no zeros, one zero, or two zeros.

The vertex or turning point in the graph of a quadratic function with only one zero lies on the x-axis.

$$f(x) = k(x - s)(x - t)$$

In this case, s and t are equal, so

$$f(x) = k(x - s)(x - s) \text{ or } f(x) = k(x - s)^2$$



A cubic function will always have at least one zero.

The function with three zeros is $f(x) = k(x - s)(x - t)(x - u)$.

The function with two zeros is $f(x) = k(x - s)(x - t)^2$, where t is the zero that is also a turning point of the function.

The function with only one zero is $f(x) = k(x - s)^3$.

CHECK, CONSOLIDATE, COMMUNICATE

1. A cubic polynomial function is $g(x) = 3(x - 2)(x + 4)(x + 6)$. State the zeros and describe the shape of the curve.
2. Explain how to describe the graph of a polynomial function.
3. Sketch a possible polynomial function for each case. Label the x-intercepts and any turning points.
   (a) f is a quadratic function where $f(x) = k(x - s)(x - t)$ and $k < 0$
   (b) f is a cubic function where $f(x) = k(x - s)(x - t)(x - u)$ and $k > 0$
   (c) f is a cubic function where $f(x) = k(x - s)(x - t)^2$ and $k < 0$
   (d) f is a quadratic function where $f(x) = k(x - s)^2$ and $k > 0$

KEY IDEAS

- The function f is a polynomial function if $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, $a_1, \ldots, a_n$ are real numbers and n is a natural number. The degree of the function is n if the leading coefficient, $a_n \neq 0$.
- The graph of a polynomial function of degree 1 is a straight line. The graph of a degree-2 polynomial is a U-shaped parabola. The graphs of many degree-3 polynomials are S-shaped curves.
- The zeros of a polynomial function $f(x)$ correspond to the x-intercepts in the graph of $f(x)$. Use the x-intercepts and several other points to graph $f(x)$.

Function	Type	Zeros	Description of Graph
$f(x) = k$	linear, degree 0	None, $k \neq 0$	horizontal line
$f(x) = k(x - s)$	linear, degree 1	$x = s$	$k > 0$, $f(x)$ is increasing $k < 0$, $f(x)$ is decreasing
$f(x) = k(x - s)(x - t)$	quadratic, degree 2	$x = s$ $x = t$	$k > 0$, $f(x)$ is decreasing and then increasing $k < 0$, $f(x)$ is increasing and then decreasing
$f(x) = k(x - s)(x - t)(x - u)$	cubic, degree 3	$x = s$ $x = t$ $x = u$	$k > 0$, $f(x)$ is increasing, decreasing, and then increasing $k < 0$, $f(x)$ is decreasing, increasing, and then decreasing

- Use the zeros of a polynomial function to find the equation of the function if one point that satisfies the function is known.

1.1 Exercises

A

1. Expand and simplify. Express each equation in standard form.
   (a) $f(x) = (x - 1)(x + 3)(x - 5)$ (b) $f(x) = (2x + 3)(x + 1)(3x - 1)$
   (c) $f(x) = -3(x - 2)(x + 3)(1 - x)$ (d) $f(x) = 5(4x + 3)(x - 5)$
   (e) $f(x) = (2x - 1)(x + 2)(3x + 1)(x - 3)$
   (f) $f(x) = -6(2x - 3)^2(x + 2)$

2. For each function in question 1, state the degree of the polynomial and identify the type of function.

3. Determine the zeros of each function.

(a) $f(x) = (x + 2)(x - 3)$ (b) $f(x) = (3x - 5)(2x + 7)$
(c) $f(x) = (x - 4)(x + 3)(x - 5)$ (d) $f(x) = (2x - 9)(3x + 4)(4x - 1)$
(e) $f(x) = (6 - 5x)(4 + 3x)(5 - 2x)$ (f) $f(x) = (x + 3)(x - 3)(2x - 5)(4 - 3x)$

4. The volume, V, of a particular box is a function of its height, h, in centimetres. If $V(h) = 4h^3 - 6h^2 + 80$, what is the volume, in cubic centimetres, when $h = 3$ cm?

5. The volume, V, of a spherical cell is a function of the radius, r, $V(r) = \frac{4}{3}\pi r^3$. What is the volume when the radius is 6.0×10^{-6} m?

6. Graph each function using its zeros and a table of values.

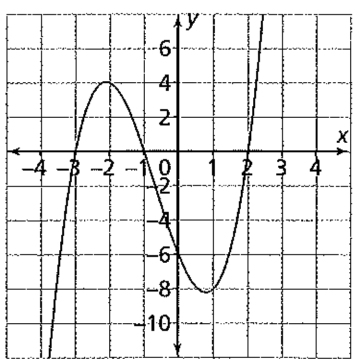
(a) $f(x) = 3(x - 3)(x + 5)$ (b) $f(x) = -2(x + 1)(x + 7)$
(c) $f(x) = 0.4(x - 1)(x + 1)(x + 2)$ (d) $f(x) = -0.2(3x - 4)(2x + 5)(2x - 3)$
(e) $f(x) = 0.5(x + 2)(x - 3)^2$ (f) $f(x) = -0.1(x - 2)^2 (x + 3)^2$

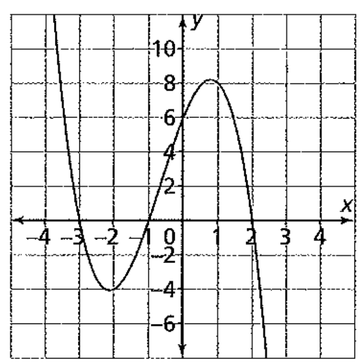
7. Without graphing, indicate whether the graph of each function is a straight line, a U-shaped parabola, or an S-shaped curve.

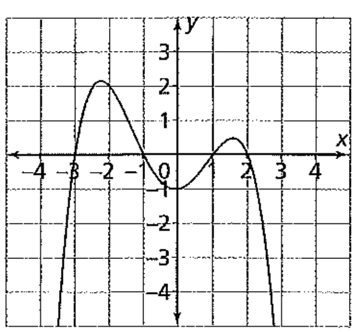
(a) $f(x) = 6 - 2x$ (b) $f(x) = 3x^2 - 2x + 1$
(c) $f(x) = -3x^3 + 6x^2 - x + 1$ (d) $f(x) = -2(x - 4)(x + 5)(x - 2)$
(e) $f(x) = -2(3x - 4)(2x + 5)$ (f) $f(x) = -5(x + 7)$

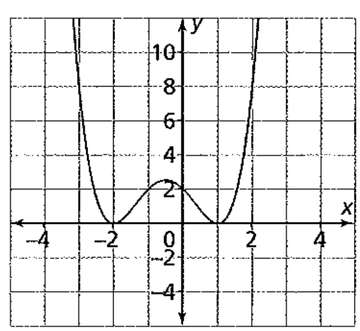
8. Confirm your answers for question 7 using graphing technology.

9. For each graph, estimate the interval on which the function is increasing, the interval on which the function is decreasing, the coordinates of all turning points, and the local maximum and minimum values.

(a)


(b)


(c)


(d)


10. For each function, use graphing technology to estimate the intervals of increase and decrease and the local maximum and minimum.

(a) $f(x) = -2(x + 3)(x - 4)$ (b) $f(x) = 3(2x - 1)(4x + 5)$

(c) $f(x) = 2(x + 4)(x - 3)(x + 5)$ (d) $f(x) = -0.5(x - 1)(x + 1)(x - 2)$

(e) $f(x) = -3(x - 5)^2$ (f) $f(x) = 1.5(x + 2)(x - 3)^2$

11. **Knowledge and Understanding:** Determine the equation of each function and sketch its graph.

(a) $f(x) = k(x + 4)(x - 6)$, and passes through $(1, -50)$

(b) $f(x) = k(x - 3)(x + 4)(x - 5)$, and passes through $(2, 18)$

(c) $f(x) = k(2x - 3)(x + 4)(3x - 5)(x - 10)$, and passes through $(1, 180)$

B

12. **Communication:** A cubic function of degree 3 can have three zeros.

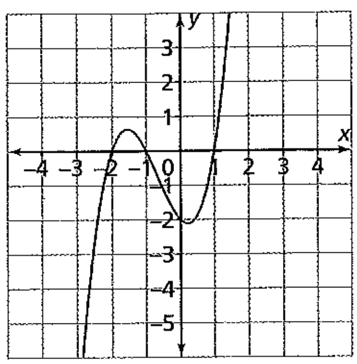
(a) Explain how to use this fact to graph the function.

(b) Graph $f(x) = k(x - 2)(x - 1)(x + 1)$ for $k = 2$ and $k = -2$.

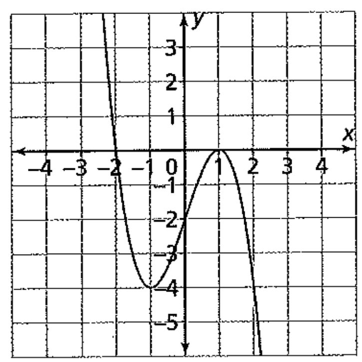
13. Verify your sketches for question 12 using a graphing calculator and comment on their closeness of fit to the actual curves.

14. Determine the equation of each cubic polynomial function from its graph.

(a)

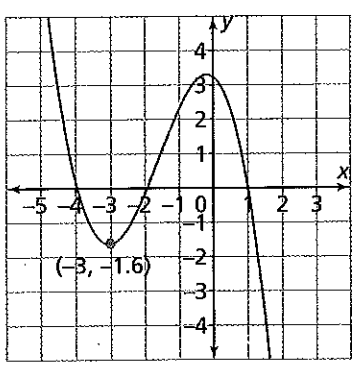


(b)

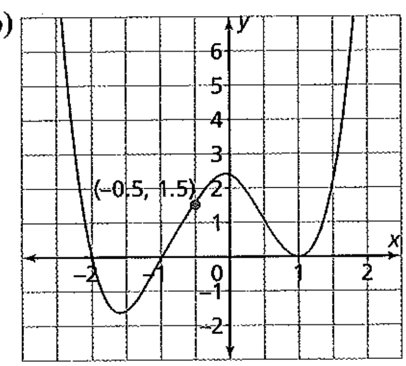


15. Determine the equation of each polynomial function.

(a)



(b)



16. Determine the quadratic function with zeros -3 and -5 and $f(7) = -720$.

17. Determine the cubic function with zeros -2, 3, and 4, and $f(5) = 28$.

18. **Application:** The function $y = 26.55x^3 - 170.56x^2 + 249.62x + 1257.80$, $0 \le x \le 4$, approximates the number of petajoules of energy used in Canada for residential purposes from 1995 to 1999.

(a) State the restricted domain of this model and describe what x represents in this case.

(b) Use graphing technology to graph the function. Describe its shape and identify the coordinates of any turning points. Use CALC and the minimum/maximum feature to help determine the intervals on which the function is increasing and decreasing.

(c) Determine the amount of residential energy Canadians used in 1998.

(d) Suppose no restrictions were placed on the domain of the function. How much residential energy would be used in 2010?

19. Let $f(x) = x^3 - 2x^2 - 5x + 6$.

(a) Use graphing technology to graph $f(x)$.

(b) Describe the behaviour of the graph in terms of the intervals when $f(x)$ is increasing and decreasing.

(c) State the value of the independent variable for all turning points.

(d) State the coordinates of all local maximum and minimum points.

20. Sure Grip Athletic Shoes tracks the relation between total sales and the number of dollars spent on a new advertising campaign. The function $S(x) = -0.000\,025x^3 + 0.015x^2$, $50 < x < 350$, represents the sales, S, in hundreds of thousands of dollars. The budget, x, for the advertising campaign is measured in tens of thousands of dollars.

(a) What advertising budget was used to generate the model? What does that mean for sales outside this domain?

(b) Use graphing technology to graph the function.

(c) What are the maximum sales this advertising campaign might generate under this model? When does this occur? What happens to sales after this point?

(d) What happens to sales when \$6 000 000 is spent on the new advertising campaign?

21. **Thinking, Inquiry, Problem Solving:** The function $f(x) = kx^3 - 8x^2 - x + 3k + 1$ has a zero, 2. Determine the value of k. Graph $f(x)$ and determine all zeros. Rewrite $f(x)$ in factored form.

22. **Check Your Understanding:** Give an example of a polynomial function in factored form for each case.

(a) degree 2; two zeros, the function decreases then increases

(b) degree 3; three zeros, the function increases, decreases, then increases

(c) degree 3; two zeros, the function decreases, increases, then decreases

(d) degree 1; one zero, the function increases

(C) 23. (a) The graph of a quadratic function that neither crosses nor touches the x-axis has no real zeros. Determine the equation of the quadratic function with roots $2 + i$ and $2 - i$.

(b) Complex roots are always expressed as conjugate pairs. Determine the equation of a quartic function with complex roots $3 + 2i$ and $2 - 3i$.

24. Let $f(x)$ be any polynomial function. Prove that the slope of the line joining points $(x_1, f(x_1))$ to $(x_2, f(x_2))$ will be positive if $f(x)$ is increasing and negative if $f(x)$ is decreasing.

René Descartes (1596–1650)

René Descartes gave his name to an important creation in mathematics, the Cartesian plane. He is also famous for saying "I think, therefore I am." Do some research—why was this saying important in philosophy?

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

Knowledge and Understanding: Determine the zeros of $p(x) = (x^2 - x - 2)(x^2 + 2x - 15)$ and find sufficient points on $p(x)$ to sketch the graph. Use your graph to estimate the maximum and minimum values.

Application: The price of equipment, in thousands of dollars, is fixed. The number of weeks of labour is w, and the production, p, in thousands of dollars is $p(w) = 6w^5 - 15w^4 - 10w^3 + 30w^2 + 10$, where $0 \le w \le 3$. Use graphing technology to graph the function. Use [TRACE] to determine the interval, in weeks, during which production is increasing and decreasing. Use [TRACE] to determine the coordinates of the turning points and end points. For what number of weeks is production at a maximum? a minimum?

Thinking, Inquiry, Problem Solving: Graph each equation on the same set of axes: $y = x^3$, $y = x^3 + x^2$, $y = x^3 + x^2 + x$. How are the graphs similar? different? Use the above results to describe the similarities and differences of $y = x^4$, $y = x^4 + x^3$, $y = x^4 + x^3 + x^2$, and $y = x^4 + x^3 + x^2 + x$, without drawing their respective graphs.

Communication: Graph a polynomial function that decreases twice and increases once.

The Chapter Problem

Developing a Model for Canada's Population

CP1. Create a scatter plot. Draw a degree-1 polynomial to represent the curve of best fit for the data. On the same graph, draw a nonlinear polynomial curve of best fit. What degree do you think the function of the curve could be? Explain.

CP2. What is the restricted domain of each model? Within this domain are the models increasing or decreasing?

CP3. Use each model to determine the population in 2001. Which model is better, based on actual data?

CP4. Use each model to determine the population in 2006 and 2016. Which model might be better? Explain.