1.8 Using Transformations to Graph Functions of the Form $y = af[k(x - d)] + c$

GOAL

Apply combinations of transformations, in a systematic order, to sketch graphs of functions.

YOU WILL NEED

- graph paper or graphing calculator

LEARN ABOUT the Math

Neil wants to sketch the graph of $f(x) = -3\sqrt{2(x + 4)} - 1$.

? How can Neil apply the transformations necessary to sketch the graph?

EXAMPLE 1 Applying a combination of transformations

Sketch the graph of $f(x) = -3\sqrt{2(x + 4)} - 1$. State the domain and range of the transformed function.

Neil's Solution

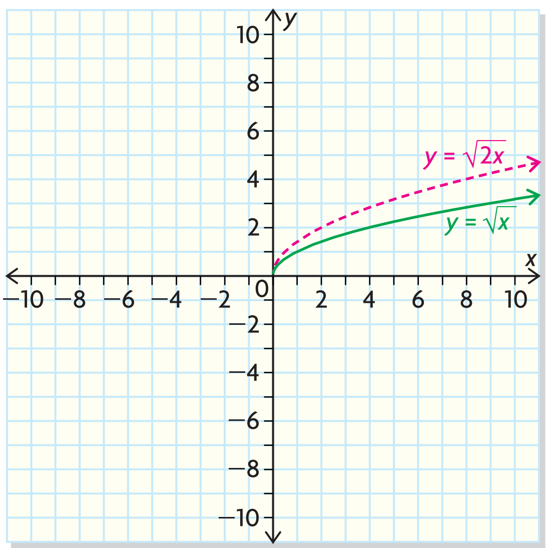
The parent function is $f(x) = \sqrt{x}$.

The function is a transformed square root function.

$$f(x) = -3\sqrt{2(x + 4)} - 1$$

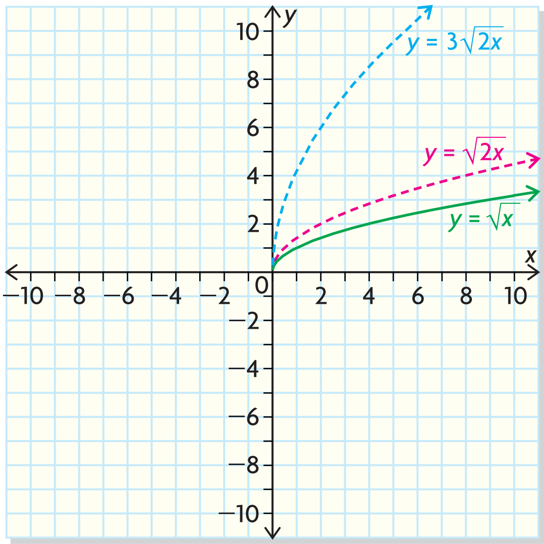
- Vertical stretch by a factor of 3
- Horizontal translation 4 units left
- Vertical translation 1 unit down
- Reflection in the *x*-axis
- Horizontal compression by a factor of $\frac{1}{2}$

I looked at each part of the function and wrote down all the transformations I needed to apply.



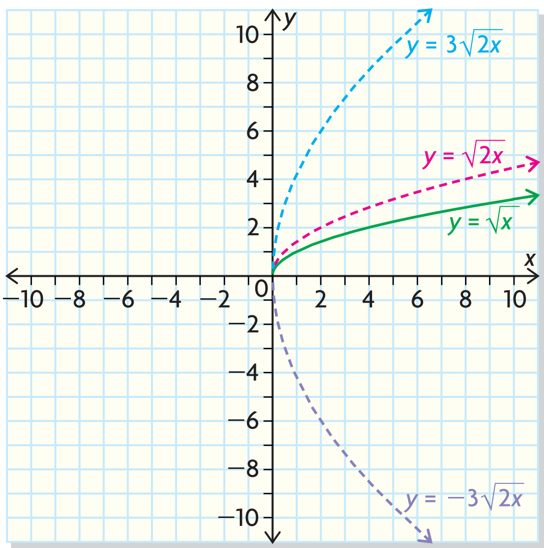
First I divided the *x*-coordinates of points on $y = \sqrt{x}$ by 2 to compress the graph horizontally by the factor $\frac{1}{2}$.

$f(x)$	$f(2x)$
(0, 0)	(0, 0)
(1, 1)	$\left(\frac{1}{2}, 1\right)$
(4, 2)	(2, 2)
(9, 3)	$\left(4\frac{1}{2}, 3\right)$



I multiplied the y-coordinates of $y = \sqrt{2x}$ by 3 to stretch the graph vertically by the factor 3.

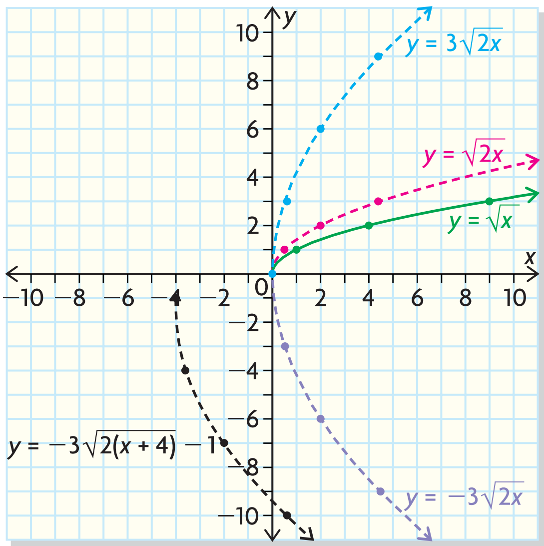
$f(x)$	$f(2x)$	$3f(2x)$
(0, 0)	(0, 0)	(0, 0)
(1, 1)	$\left(\frac{1}{2}, 1\right)$	$\left(\frac{1}{2}, 3\right)$
(4, 2)	(2, 2)	(2, 6)
(9, 3)	$\left(4\frac{1}{2}, 3\right)$	$\left(4\frac{1}{2}, 9\right)$



I flipped the graph of $y = 3\sqrt{2x}$ over the x-axis.

$f(x)$	$f(2x)$	$3f(2x)$	$-3f(2x)$
(0, 0)	(0, 0)	(0, 0)	(0, 0)
(1, 1)	$\left(\frac{1}{2}, 1\right)$	$\left(\frac{1}{2}, 3\right)$	$\left(\frac{1}{2}, -3\right)$
(4, 2)	(2, 2)	(2, 6)	(2, −6)
(9, 3)	$\left(4\frac{1}{2}, 3\right)$	$\left(4\frac{1}{2}, 9\right)$	$\left(4\frac{1}{2}, -9\right)$

Translate the graph 4 units left and 1 unit down.



I did both shifts together. I subtracted 4 from each of the x-coordinates and subtracted 1 from each of the y-coordinates of the graph of $y = -3\sqrt{2x}$.

$f(x)$	$f(2x)$	$3f(2x)$	$-3f(2x)$	$-3f(2(x + 4)) - 1$
(0, 0)	(0, 0)	(0, 0)	(0, 0)	(−4, −1)
(1, 1)	$\left(\frac{1}{2}, 1\right)$	$\left(\frac{1}{2}, 3\right)$	$\left(\frac{1}{2}, -3\right)$	$\left(-3\frac{1}{2}, -4\right)$
(4, 2)	(2, 2)	(2, 6)	(2, −6)	(−2, −7)
(9, 3)	$\left(4\frac{1}{2}, 3\right)$	$\left(4\frac{1}{2}, 9\right)$	$\left(4\frac{1}{2}, -9\right)$	$\left(\frac{1}{2}, -10\right)$

Domain $= \{x \in \mathbf{R} \mid x \geq -4\}$

Range $= \{y \in \mathbf{R} \mid y \leq -1\}$

From the final graph, $x \geq -4$ and $y \leq -1$.

Reflecting

A. How do the numbers in the function $f(x) = -3\sqrt{2(x + 4)} - 1$ affect the x- and y-coordinates of each point on the parent function ?

B. How did Neil determine the domain and range of the final function?

C. How does the order in which Neil applied the transformations compare with the order of operations for numerical expressions?

D. Sarit says that she can graph the function in two steps. She would do both stretches or compressions and any reflections to the parent function first and then both translations. Do you think this will work? Explain.

APPLY the Math

EXAMPLE 2 Applying transformations to the equation and the graph

Some transformations are applied, in order, to the reciprocal function $f(x) = \frac{1}{x}$:

- horizontal stretch by the factor 3
- vertical stretch by the factor 2
- reflection in the y-axis
- translation 5 units right and 4 units up

a) Write the equation for the final transformed function $g(x)$.
b) Sketch the graphs of $f(x)$ and $g(x)$.
c) State the domain and range of both functions.

Lynn's Solution

a)
$$g(x) = af[k(x - d)] + c$$
$$= 2f\left[-\frac{1}{3}(x - 5)\right] + 4$$
$$= \frac{2}{\left(-\frac{1}{3}(x - 5)\right)} + 4$$

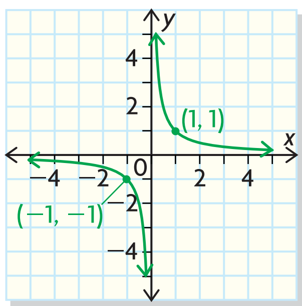
I built up the equation from the transformations.

A horizontal stretch by the factor 3 and a reflection in the y-axis means that $k = -\frac{1}{3}$.

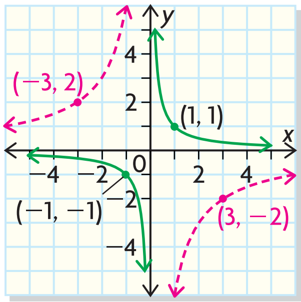
$a = 2$, because there is a vertical stretch by the factor 2.

$d = 5$ and $c = 4$, because the translation is 5 units right and 4 units up.

b) Graph of $f(x)$:



I sketched the graph of $f(x)$ and labelled the points (1, 1) and $(-1, -1)$. The vertical asymptote is $x = 0$ and the horizontal asymptote is $y = 0$.

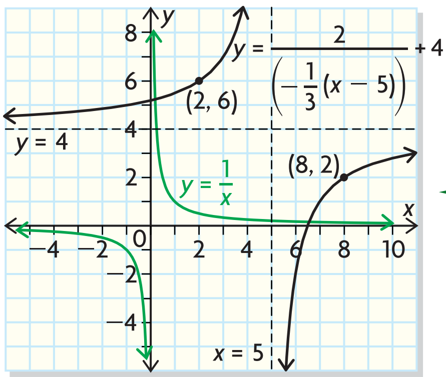


I applied the stretches and reflection to the labelled points by multiplying the x-coordinates by -3 and the y-coordinates by 2.

$(1, 1)$ became $(-3, 2)$ and $(-1, -1)$ became $(3, -2)$. The asymptotes did not change, since x and y still couldn't be 0.

I made a sketch of the stretched and reflected graph before applying the translation.

Graph of $g(x)$:



To apply the translations, 5 right and 4 up, I drew in the translated asymptotes first.

Since all the points moved 5 right, the new vertical asymptote is $x = 5$.

Since all the points moved up 4, the new horizontal asymptote is $y = 4$.

Then I drew the stretched and reflected graph in the new position after the translation.

I labelled the graphs and wrote the equations for the asymptotes.

c) For $f(x)$,

Domain $= \{x \in \mathbf{R} \mid x \neq 0\}$

Range $= \{y \in \mathbf{R} \mid y \neq 0\}$

For $g(x)$,

Domain $= \{x \in \mathbf{R} \mid x \neq 5\}$

Range $= \{y \in \mathbf{R} \mid y \neq 4\}$

I used the equations of the asymptotes to help determine the domain and range.

The graphs do not meet their asymptotes, so for $f(x)$, x cannot be 0 and y cannot be 0. Also, for $g(x)$, x cannot be 5 and y cannot be 4.

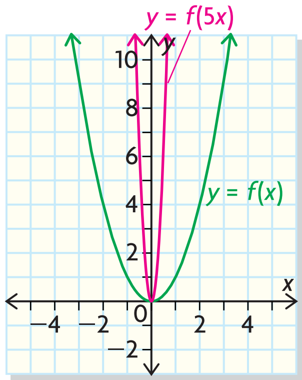
EXAMPLE 3 Factoring out k before applying transformations

For $f(x) = x^2$, sketch the graph of $g(x) = f(-5x + 10)$.

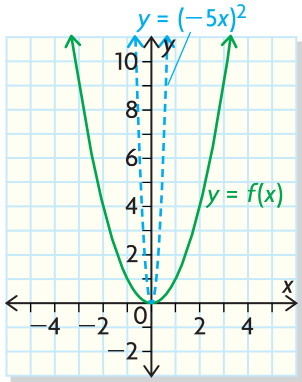
Stefan's Solution

$$g(x) = f(-5x + 10)$$
$$= f[-5(x - 2)]$$

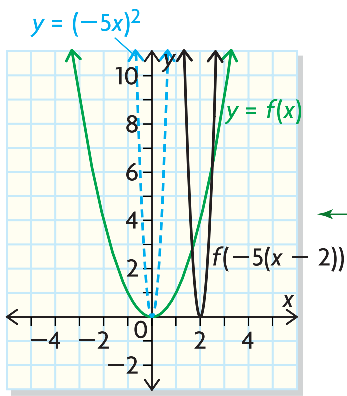
I wrote $g(x)$ in $af[k(x - d)] + c$ form by factoring out $k = -5$.



I graphed $y = f(x)$ and compressed the graph horizontally by the factor $\frac{1}{5}$. This gave me the graph of $y = f(5x)$.



I reflected $y = f(5x)$ in the y-axis. The graph of $y = f(-5x)$ looked the same because the y-axis is the axis of symmetry for $y = f(5x)$.



I translated the compressed and reflected graph 2 units right. This gave me the graph of $y = f[-5(x - 2)]$.

EXAMPLE 4 Identifying the equation of a transformed function from its graph

Match each equation to its graph. Explain your reasoning.

1. $y = \dfrac{1}{0.3(x+1)} - 2$

2. $y = -4|x+2| + 1$

3. $y = -\sqrt{3(x+2)} - 1$

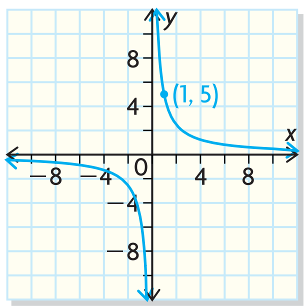
4. $y = \sqrt{-0.4(x-4)} + 3$

5. $y = (0.5(x-4))^2 + 2$

6. $y = \dfrac{5}{x}$

7. $y = -3(x+1)^2 + 4$

A



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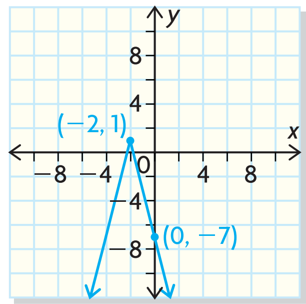

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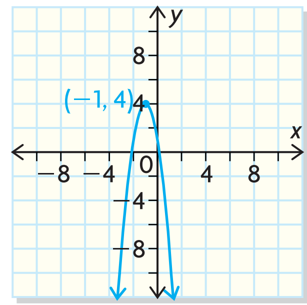

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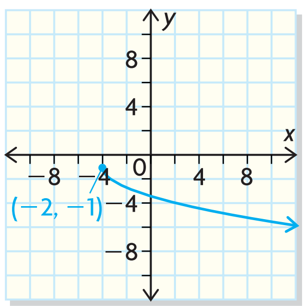

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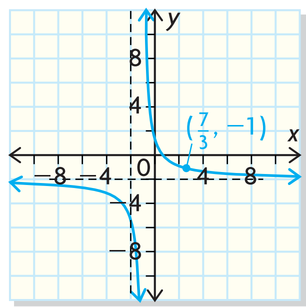

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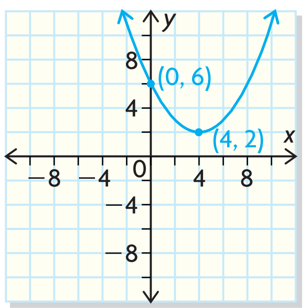

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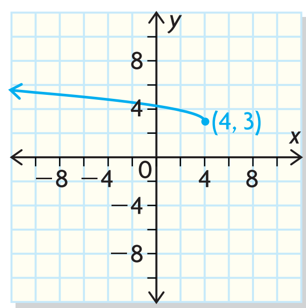


Donna's Solution

Graph A matches equation 6.

Graph A is like the graph of $y = \frac{1}{x}$, but it has been stretched vertically. The point (1, 1) has been stretched to (1, 5), so the scale factor is 5. The equation really is $y = \frac{5}{x}$.

Graph B matches equation 3.

This is the graph of a square root function that has been flipped over the x-axis, so, in the equation, a will be negative.

The parent square root graph has been compressed horizontally or stretched vertically. It starts at $(-2, -1)$ instead of (0, 0), so it has been translated 2 units left and 1 unit down. So, $c = -2$ and $d = -1$.

Graph C matches equation 5.

Graph C is a parabola, so it has to match equation 5 or equation 7. Since $a > 0$, the parabola opens upward, so the answer can't be equation 7 and has to be equation 5.

I checked: The vertex is (4, 2), so $d = 4$ and $c = 2$.

Graph C is wider than the parent function, so it has been stretched horizontally or compressed vertically. Equation 5 is the equation of a parabola with vertex (4, 2), that opens up, and that has been stretched horizontally by the factor $\frac{1}{0.5} = 2$.

Graph D matches equation 2.

This is the graph of an absolute value function. The parent graph has been reflected in the x-axis, stretched vertically, and shifted 2 units left and 1 unit up. The equation must have $a < -1$, $d = -2$, and $c = 1$.

Graph E matches equation 1.

This is a transformation of the graph of $y = \frac{1}{x}$, so the answer has to be equation 1 or equation 6. The equations for the asymptotes are $x = -1$ and $y = -2$, so $d = -1$ and $c = -2$. This matches equation 1.

Also, in the equation, $k = 0.3$ means that the parent graph has been stretched horizontally by the factor $\frac{1}{0.3}$. The point (1, 1) on the parent graph becomes $\left(\frac{10}{3}, 1\right)$, when you multiply the x-coordinate by $\frac{1}{0.3}$. Then, this point becomes $\left(\frac{7}{3}, -1\right)$, when you apply the translations by subtracting 1 from the x-coordinate and 2 from the y-coordinate.

Graph F matches equation 4.

This is another square root function. The parent function has been flipped over the y-axis, so $k < 0$. It has been stretched horizontally, so $-1 < k < 0$, and translated 4 units right and 3 units up, so $d = 4$ and $c = 3$.

Graph G matches equation 7.

Graph G is a parabola that opens down, so it has to match equation 7 because it is vertically stretched (narrow) and has vertex at $(-1, 4)$. Equation 7 has $a = -3$, which means that the parabola opens down and is vertically stretched by the factor 3. Also, $c = -1$ and $d = 4$, which means that the vertex is $(-1, 4)$, as in graph G.

In Summary

Key Ideas

- You can graph functions of the form $g(x) = af[k(x - d)] + c$ by applying the appropriate transformations to the key points of the parent function, one at a time, making sure to apply a and k before c and d. This order is like the order of operations for numerical expressions, since multiplications (stretches, compressions, and reflections) are done before additions and subtractions (translations).
- When using transformations to graph, you can apply a and k together, then c and d together, to get the desired graph in fewer steps.

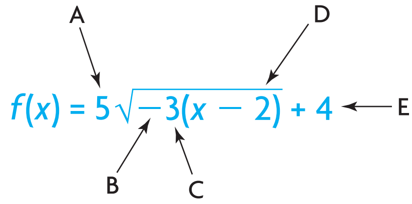
Need to Know

- The value of a determines the vertical stretch or compression and whether there is a reflection in the x-axis:
  - When $|a| > 1$, the graph of $y = f(x)$ is stretched vertically by the factor $|a|$.
  - For $0 < |a| < 1$, the graph is compressed vertically by the factor $|a|$.
  - When $a < 0$, the graph is also reflected in the x-axis.
- The value of k determines the horizontal stretch or compression and whether there is a reflection in the y-axis:
  - When $|k| > 1$, the graph is compressed horizontally by the factor $\frac{1}{|k|}$.
  - When $0 < |k| < 1$, the graph is stretched horizontally by the factor $\frac{1}{|k|}$.
  - When $k < 0$, the graph is also reflected in the y-axis.
- The value of d determines the horizontal translation:
  - For $d > 0$, the graph is translated d units right.
  - For $d < 0$, the graph is translated d units left.
- The value of c determines the vertical translation:
  - For $c > 0$, the graph is translated c units up.
  - For $c < 0$, the graph is translated c units down.

CHECK Your Understanding

1. Use words from the list to describe the transformations indicated by the arrows.

horizontal	x-axis
vertical	y-axis
stretch	factor
compression	up
reflection	down
translation	right
	left



2. Match each operation to one of the transformations from question 1.

Divide the x-coordinates by 3.	A
Multiply the y-coordinates by 5.	B
Multiply the x-coordinates by -1.	C
Add 4 to the y-coordinate.	D
Add 2 to the x-coordinate.	E

3. Complete the table for the point $(1, 1)$.

$f(x)$	$f(3x)$	$f(-3x)$	$5f(-3x)$	$5f(-3(x-2)) + 4$
(1, 1)				

PRACTISING

4. **K** Explain what transformations you would need to apply to the graph of $y = f(x)$ to graph each function.

a) $y = 3f(x) - 1$ **c)** $y = f(2x) - 5$ **e)** $y = \frac{2}{3}f(x+3) + 1$

b) $y = f(x-2) + 3$ **d)** $y = -f\left(\frac{1}{2}x\right) - 2$ **f)** $y = 4f(-x) - 4$

5. Sketch each set of functions on the same set of axes.

a) $y = x^2, y = 3x^2, y = 3(x-2)^2 + 1$

b) $y = \sqrt{x}, y = \sqrt{3x}, y = \sqrt{-3x}, y = \sqrt{-3(x+1)} - 4$

c) $y = \frac{1}{x}, y = \frac{2}{x}, y = -\frac{2}{x}, y = -\frac{2}{x-1} + 3$

d) $y = |x|, y = \left|\frac{1}{2}x\right|, y = -\left|\frac{1}{2}x\right|, y = -\left|\frac{1}{2}(x+3)\right| - 2$

6. Explain what transformations you would need to apply to the graph of $y = f(x)$ to graph each function.

a) $y = f\left(\frac{1}{3}(x+4)\right)$ **c)** $y = -3f(2(x-1)) - 3$

b) $y = 2f(-(x-3)) + 1$

7. If $f(x) = x^2$, sketch the graph of each function and state the domain and range.

a) $y = f(x - 2) + 3$

b) $y = -f\left(\frac{1}{4}(x + 1)\right) + 2$

c) $y = 0.5f(3(x - 4)) - 1$

8. If $f(x) = \sqrt{x}$, sketch the graph of each function and state the domain and range.

a) $y = f(x - 1) + 4$

b) $y = f\left(-\frac{1}{2}(x + 4)\right) - 3$

c) $y = -2f(-(x - 2)) + 1$

9. If $f(x) = |x|$, sketch the graph of each function and state the domain and range.

a) $y = 2f(x - 3)$

b) $y = 4f(2(x - 1)) - 2$

c) $y = -\frac{1}{2}f(3(x + 2)) + 4$

10. Describe the transformations that you would apply to the graph of $f(x) = \frac{1}{x}$ to transform it into each of these graphs.

a) $y = \frac{1}{x - 2}$

b) $y = \frac{1}{x} + 2$

c) $y = 0.5\left(\frac{1}{x}\right)$

d) $y = \frac{2}{x}$

e) $y = \frac{1}{2x}$

f) $y = -\frac{1}{x}$

11. For $f(x) = x^2$, sketch the graph of $g(x) = f(2x + 6)$.

12. For $f(x) = \sqrt{x}$, sketch the graph of $h(x) = f(-3x - 12)$.

13. For $f(x) = |x|$, sketch the graph of $p(x) = f(4x + 8)$.

14. **A** Low and high blood pressure can both be dangerous. Doctors use a special index, P_d, to measure how far from normal someone's blood pressure is. In the equation $P_d = |P - \overline{P}|$, P is a person's systolic blood pressure and $\overline{P}$ is the normal systolic blood pressure. Sketch the graph of this index. Assume that normal systolic blood pressure is 120 mm(Hg).

15. Bhavesh uses the relationship $\text{Time} = \frac{\text{Distance}}{\text{Speed}}$ to plan his kayaking trips. Tomorrow Bhavesh plans to kayak 20 km across a calm lake. He wants to graph the relation $T(s) = \frac{20}{s}$ to see how the time, T, it will take varies with his kayaking speed, s. The next day, he will kayak 15 km up a river that flows at 3 km/h. He will need the graph of $T(s) = \frac{15}{s - 3}$ to plan this trip. Use transformations to sketch both graphs.

16. The graph of $g(x) = \sqrt{x}$ is reflected across the y-axis, stretched vertically by the factor 3, and then translated 5 units right and 2 units down. Draw the graph of the new function and write its equation.

17. The graph of $y = f(x)$ is reflected in the y-axis, stretched vertically by the factor 3, and then translated up 2 units and 1 unit left. Write the equation of the new function in terms of f.

18. Match each equation to its graph. Explain your reasoning.

a) $y = \frac{3}{-(x-2)} + 1$

b) $y = 2|x-3| - 2$

c) $y = -2\sqrt{x+3} - 2$

d) $y = (0.25(x-2))^2 - 3$

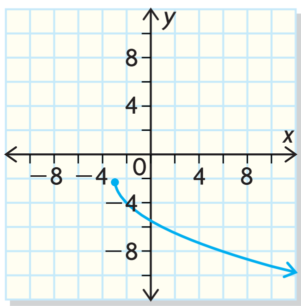
e) $y = -\frac{4}{x} - 3$

f) $y = -0.5|x+4| + 2$

g) $y = -0.5\sqrt{1-x} + 1$

h) $y = -\frac{1}{2}(x+4)^2 + 1$

A



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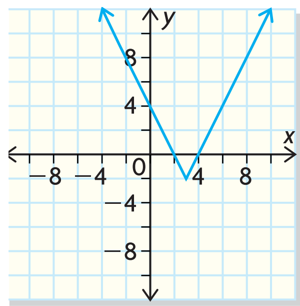

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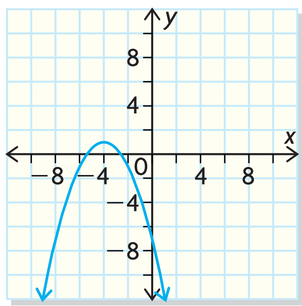

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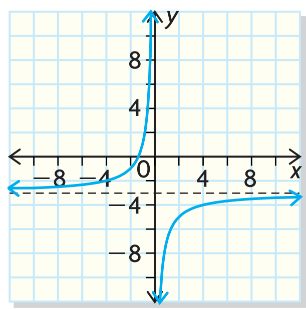

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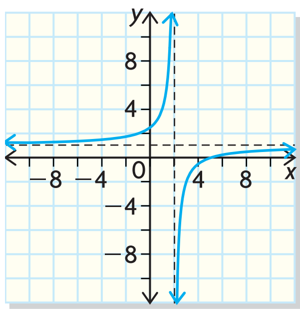

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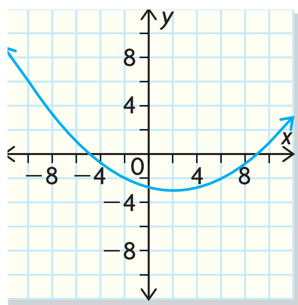

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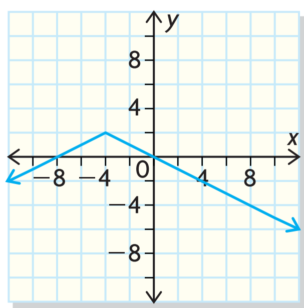

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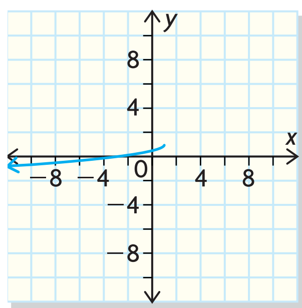


19. The function $y = f(x)$ has been transformed to $y = af[k(x - d)] + c$. Determine a, k, c, and d; sketch the graph; and state the domain and range for each transformation.

a) A vertical stretch by the factor 2, a reflection in the x-axis, and a translation 4 units right are applied to $y = \sqrt{x}$.

b) A vertical compression by the factor $\frac{1}{2}$, a reflection in the y-axis, a translation 3 units left, and a translation 4 units down are applied to $f(x) = \frac{1}{x}$.

c) A horizontal compression by the factor $\frac{1}{3}$, a vertical stretch by the factor 3, a translation 1 unit right, and a translation 6 units down are applied to $y = |x|$.

20. If $f(x) = (x - 2)(x + 5)$, determine the x-intercepts for each function.

a) $y = f(x)$

b) $y = -4f(x)$

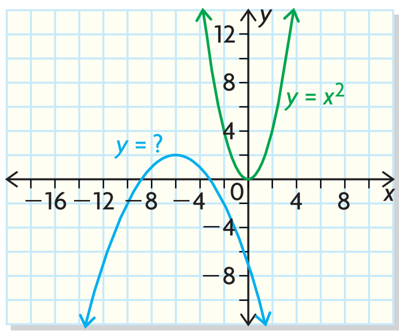
c) $y = f\left(-\frac{1}{3}x\right)$

d) $y = f(-(x + 2))$

21. C List the steps you would take to sketch the graph of a function of the form $y = af(k(x - d)) + c$ when $f(x)$ is one of the parent functions you have studied in this chapter. Discuss the roles of a, k, d, and c, and the order in which they would be applied.

Extending

22. The graphs of $y = x^2$ and another parabola are shown.



a) Determine a combination of transformations that would produce the second parabola from the first.

b) Determine a possible equation for the second parabola.

23. Compare the graphs and the domains and ranges of $f(x) = x^2$ and $g(x) = \sqrt{x}$. How are they alike? How are they different? Develop a procedure to obtain the graph of $g(x)$ from the graph of $f(x)$.