1.5 The Inverse Function and Its Properties

YOU WILL NEED

- graph paper
- Mira™ (transparent mirror) (optional)

GOAL

Determine inverses of linear functions and investigate their properties.

INVESTIGATE the Math

The Backyard Paving Company charges $10/sq ft for installing interlocking paving stones, plus a $50 delivery fee. The company calculates the cost to the customer as a function of the area to be paved. Tom wants to express area in terms of cost to see how much of his yard he can pave for different budget amounts.

A

x Area (sq ft)	y Cost ($)
40	450
80	
120	
160	
200	

? What relation can Tom use, and how is it related to the function used by the company?

A. Copy and complete table A, using the company's prices. What is the independent variable in table A? the dependent variable?

B. Is the relation in table A a function? Explain.

C. Write the equation for $f(x)$ that describes the cost as a function of area.

D. Graph $f(x)$. Use the same scale of -100 to 2100 on each axis.

E. Tom needs to do the reverse of what the company's function does. Copy and complete table E for Tom. What is the independent variable? the dependent variable? How does this table compare with table A?

E

Cost ($)	Area (sq ft)
450	40
850	
1250	
1650	
2050	

F. The relationship in part E is the inverse of the cost function. Graph this inverse relation on the same axes as those in part D. Is this relation a function? Explain.

inverse of a function

the reverse of the original function; undoes what the original function has done

G. Draw the line $y = x$ on your graph. Place a Mira along the line $y = x$, or fold your graph paper along that line. What do you notice about the two graphs? Where do they intersect?

H. Compare the coordinates of points that lie on the graph of the cost function with those which lie on the graph of its inverse. What do you notice?

I. Write the slopes and y-intercepts of the two lines.
i) How are the slopes related?
ii) How are the y-intercepts related?
iii) Use the slope and y-intercept to write an equation for the inverse.

J. Use inverse operations on the cost function, f, to solve for x. Compare this equation with the equation of the inverse you found in part I.

K. Make a list of all the connections you have observed between the Backyard Paving Company's cost function and the one Tom will use.

Reflecting

L. How would a table of values for a linear function help you determine the inverse of that function?

M. **i)** How can you determine the coordinates of a point on the graph of the inverse function if you know a point on the graph of the original function?
ii) How could you use this relationship to graph the inverse?

N. How are the domain and range of the inverse related to the domain and range of a linear function?

O. How could you use inverse operations to determine the equation of the inverse of a linear function from the equation of the function?

APPLY the Math

EXAMPLE 1 — Representing the equation of the inverse of a linear function

Find the inverse of the function defined by $f(x) = 2 - 5x$. Is the inverse a function? Explain.

Jamie's Solution: Reversing the Operations

In the equation $f(x) = 2 - 5x$, the operations on x are as follows: Multiply by -5 and then add 2.

I wrote down the operations on x in the order they were applied.

To reverse these operations, subtract 2 and then divide the result by -5.

Then I worked backward and wrote the inverse operations.

$$f^{-1}(x) = \frac{x-2}{-5} \quad \text{or}$$

$$f^{-1}(x) = -\frac{1}{5}x + \frac{2}{5}$$

I used these inverse operations to write the equation of the inverse.

The inverse is linear, so it must be a function, since all linear relations except vertical lines are functions.

I knew the inverse was a line.

Communication *Tip*

The function f^{-1} is the inverse of the function f. This use of -1 is different from raising values to the power -1.

Lynette's Solution: Interchanging the Variables

$$f(x) = 2 - 5x$$

$$y = -5x + 2$$

I wrote the function in $y = mx + b$ form by putting y in place of $f(x)$.

$$x = -5y + 2$$

I knew that if (x, y) is on the graph of $f(x)$, then (y, x) is on the inverse graph, so I switched x and y in the equation.

$$x - 2 = -5y + 2 - 2$$

$$x - 2 = -5y$$

$$\frac{x-2}{-5} = y$$

I solved for y by subtracting 2 from both sides and dividing both sides by -5.

$$f^{-1}(x) = \frac{x-2}{-5} \quad \text{or} \quad f^{-1}(x) = \frac{2-x}{5}$$

I wrote the equation in function notation.

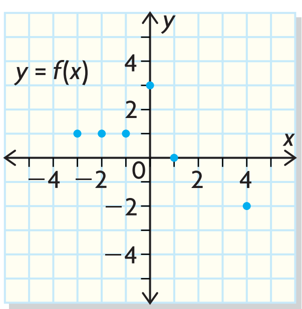
The inverse is a function.

The graph of $y = f^{-1}(x)$ is a straight line with slope $= -\frac{1}{5}$. The graph passes the vertical-line test.

EXAMPLE 2 Relating the graphs of functions and their inverses

Use the graph of each function to obtain the graph of the inverse. Is the inverse a function? Explain.

a)



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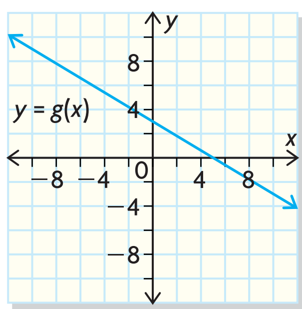


Carlos's Solution

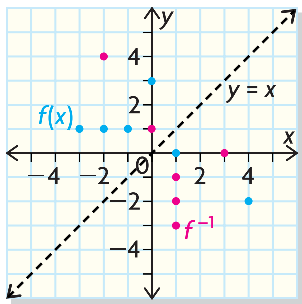
a) $f(x)$ is a function represented by the set of points $\{(-3, 1), (-2, 1), (-1, 1), (0, 3), (1, 0), (4, -2)\}$.

So $f^{-1}(x)$ is $\{(1, -3), (1, -2), (1, -1), (3, 0), (0, 1), (-2, 4)\}$.

I wrote the coordinates of the points in the graph and then switched the *x*- and *y*-coordinates of each point. That gave me the inverse.

Plot the points for the inverse and draw the line $y = x$ to check for symmetry.

I plotted the points in red.

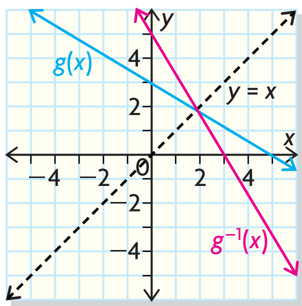


I checked that the points on one side of the line $y = x$ were mirror images of the points on the other side.

The inverse is not a function: The graph fails the vertical-line test at $x = 1$.

There are three red points for $x = 1$, so a vertical line drawn here would go through three points.

b)



I wrote the coordinates of the *x*- and *y*-intercepts of $g(x)$: (5, 0) and (0, 3).

Then I switched the coordinates to find the two points (0, 5) and (3, 0) of $g^{-1}(x)$. I noticed that they were the intercepts.

I plotted the two points of $g^{-1}(x)$ and joined them with a straight line.

I drew the line $y = x$ and checked that the graphs of $g(x)$ and $g^{-1}(x)$ crossed on that line.

The inverse is a function.

The inverse is a function because it passes the vertical-line test.

EXAMPLE 3 Using the inverse of a linear function to solve a problem

Recall from Lesson 1.2 that the temperature below Earth's surface is a function of depth and can be defined by $T(d) = 11 + 0.015d$.

a) State the domain and range of $T(d)$.
b) Determine the inverse of this function.
c) State the domain and range of $T^{-1}(d)$.
d) Explain what the inverse represents.

Erynn's Solution

a) Domain $= \{d \in \mathbf{R} \mid 0 \le d \le 5000\}$

I realized that *d* is 0 m on the surface. This is the beginning of the domain. The deeper mine has a depth of 4100 m, so I chose to end the domain at 5000.

Range $=$ $\{T(d) \in \mathbf{R} \mid 11 \le T(d) \le 86\}$

I calculated the beginning and end of the range by substituting $d = 0$ and $d = 5000$ into the equation for $T(d)$.

b)

$$T(d) = 11 + 0.015d$$
$$y = 11 + 0.015x$$

I wrote the temperature function with *y* and *x* instead of $T(d)$ and *d*.

$$x = 11 + 0.015y$$
$$x - 11 = 0.015y$$
$$\frac{x - 11}{0.015} = y$$

I switched *x* and *y* and solved for *y* to get the inverse equation.

$$d(T) = \frac{T - 11}{0.015}$$ is the inverse function.

Because I had switched the variables, I knew that *y* was now distance and *x* was temperature. I wrote the inverse in function notation.

c) Domain $\{T \in \mathbf{R} \mid 11 \le T \le 86\}$

Range $=$ $\{d(T) \in \mathbf{R} \mid 0 \le d(T) \le 5000\}$

The domain of the inverse is the same as the range of the original function, and the range of the inverse is the same as the domain of the original function.

d) The inverse shows the depth as a function of the temperature.

$$d(22) = \frac{22 - 11}{0.015}$$
$$\doteq 733$$

The inverse function is used to determine how far down a mine you would have to go to reach a temperature of, for example, 22 °C. I substituted 22 for *T* in the equation to get the answer.

When the temperature is 22 °C, the depth is about 733 m.

Someone planning a geothermal heating system would need this kind of information.

In Summary

Key Ideas

- The inverse of a linear function is the reverse of the original function. It undoes what the original has done and can be found using the inverse operations of the original function in reverse order. For example, to apply the function defined by $f(x) = 5x + 8$, multiply x by 5 and then add 8. To reverse this function, subtract 8 from x and then divide the result by 5: $f^{-1}(x) = \frac{x - 8}{5}$.
- The inverse of a function is not necessarily a function itself.

Need to Know

- A way to determine the inverse function is to switch the two variables and solve for the previously independent variable. For example, if $y = 5x + 8$, rewrite this equation as $x = 5y + 8$ and solve for y to get $y = \frac{x - 8}{5}$.
- If the original function is linear (with the exception of a horizontal line), the inverse is also a linear function.
- f^{-1} is the notation for the inverse function of f.
- If (a, b) is a point on the graph of $y = f(x)$, then (b, a) is a point on the graph of $y = f^{-1}(x)$. This implies that the domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.
- The graph of the inverse is the reflection of the graph of $y = f(x)$ in the line $y = x$.

CHECK Your Understanding

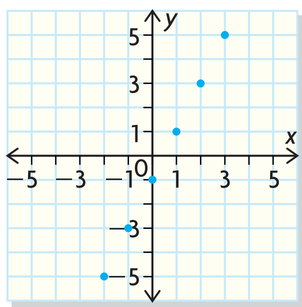
1. Determine the inverse relation for each set of ordered pairs. Graph each relation and its inverse. Which of the relations and inverse relations are functions?

a) $\{(-2, 3), (0, 4), (2, 5), (4, 6)\}$

b) $\{(2, 5), (2, -1), (3, 1), (5, 1)\}$

2. Copy the graph of each function and graph its inverse. For each graph, identify the points that are common to the function and its inverse. Which inverse relations are functions?

a)



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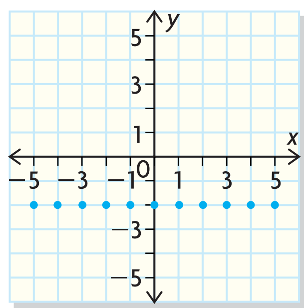

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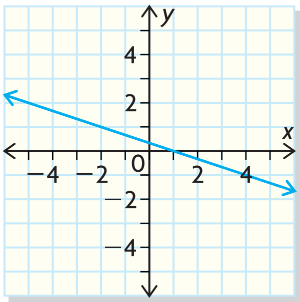

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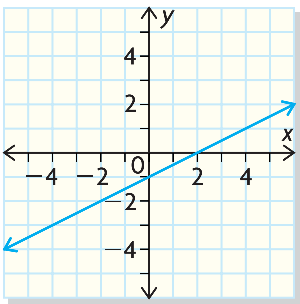

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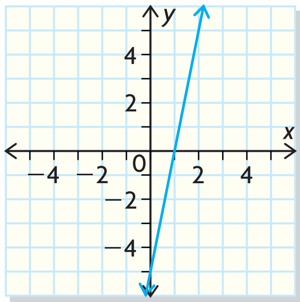

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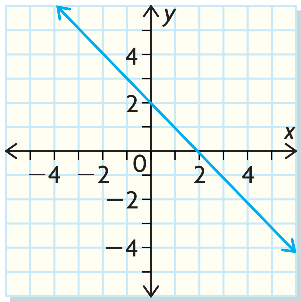


3. Determine whether each pair of functions described in words are inverses.

a) f: Multiply by 3, then add 1; g: Divide by 3, then subtract 1.

b) f: Multiply by 5, then subtract 2; g: Add 2, then divide by 5.

4. For each linear function, interchange x and y. Then solve for y to determine the inverse.

a) $y = 4x - 3$

b) $y = 2 - \frac{1}{2}x$

c) $3x + 4y = 6$

d) $2y - 10 = 5x$

PRACTISING

5. Determine the inverse of each linear function by reversing the operations.

a) $f(x) = x - 4$

b) $f(x) = 3x + 1$

c) $f(x) = 5x$

d) $f(x) = \frac{1}{2}x - 1$

e) $f(x) = 6 - 5x$

f) $f(x) = \frac{3}{4}x + 2$

6. Determine the inverse of each linear function by interchanging the variables.

a) $f(x) = x + 7$

b) $f(x) = 2 - x$

c) $f(x) = 5$

d) $f(x) = -\frac{1}{5}x - 2$

e) $f(x) = x$

f) $f(x) = \frac{x - 3}{4}$

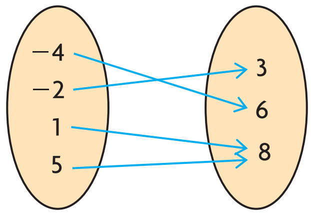
7. Sketch the graph of each function in questions 5 and 6, and sketch its inverse. Is each inverse linear? Is each inverse a function? Explain.

8. For each function, determine the inverse, sketch the graphs of the function and its inverse, and state the domain and range of both the function and its inverse. In each case, how do the domain and range of the function compare with the domain and range of the inverse?

K

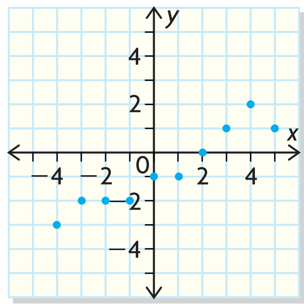
a) $\{(-1, 2), (1, 4), (2, 6), (3, 8)\}$

b)



c) $f(x) = 1 - 3x$

d)



9. **a)** Determine f^{-1} for the linear function $f(x) = 5x - 2$.
b) Graph f and f^{-1} on the same axes.
c) Explain how you can tell that f^{-1} is also a linear function.
d) State the coordinates of any points that are common to both f and f^{-1}.
e) Compare the slopes of the two lines.
f) Repeat parts (a) to (e) for $g(x) = -\frac{1}{2}x + 3$, $h(x) = 2x - 1$, $p(x) = 6 - x$, and $q(x) = 2$.

10. For $g(t) = 3t - 2$, determine each value.

a) $g(13)$
b) $g(7)$
c) $\dfrac{g(13) - g(7)}{13 - 7}$
d) $g^{-1}(13)$
e) $g^{-1}(7)$
f) $\dfrac{g^{-1}(13) - g^{-1}(7)}{13 - 7}$

11. Explain what parts (c) and (f) represent in question 10.

12. The formula for converting a temperature in degrees Celsius into degrees Fahrenheit is $F = \frac{9}{5}C + 32$. Shirelle, an American visitor to Canada, uses a simpler rule to convert from Celsius to Fahrenheit: Double the Celsius temperature, then add 30.

A

a) Use function notation to write an equation for this rule. Call the function f and let x represent the temperature in degrees Celsius.
b) Write f^{-1} as a rule. Who might use this rule?
c) Determine $f^{-1}(x)$.
d) One day, the temperature was 14 °C. Use function notation to express this temperature in degrees Fahrenheit.
e) Another day, the temperature was 70 °F. Use function notation to express this temperature in degrees Celsius.

13. Ben, another American visitor to Canada, uses this rule to convert centimetres to inches: Multiply by 4 and then divide by 10. Let the function g be the method for converting centimetres to inches, according to Ben's rule.
a) Write g^{-1} as a rule.
b) Describe a situation in which the rule for g^{-1} might be useful.

c) Determine $g(x)$ and $g^{-1}(x)$.
d) One day, 15 cm of snow fell. Use function notation to represent this amount in inches.
e) Ben is 5 ft 10 in. tall. Use function notation to represent his height in centimetres.

14. Ali did his homework at school with a graphing calculator. He determined that the equation of the line of best fit for some data was $y = 2.63x - 1.29$. Once he got home, he realized he had mixed up the independent and dependent variables. Write the correct equation for the relation in the form $y = mx + b$.

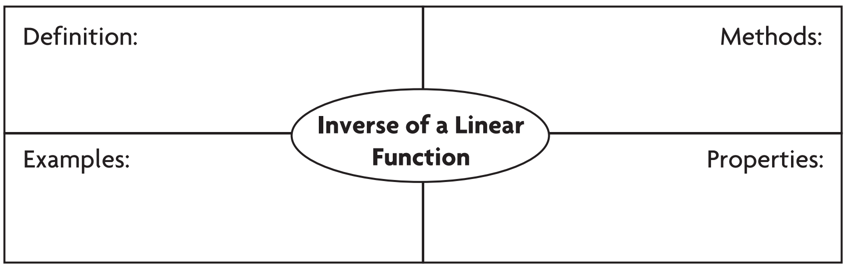
15. Tiffany is paid \$8.05/h, plus 5% of her sales over \$1000, for a 40 h work week. For example, suppose Tiffany sold \$1800 worth of merchandise. Then she would earn \$8.05(40) + 0.05(\$800) = \$362.

a) Graph the relation between Tiffany's total pay for a 40 h work week and her sales for that week.
b) Write the relation in function notation.
c) Graph the inverse relation.
d) Write the inverse relation in function notation.
e) Write an expression in function notation that represents her sales if she earned \$420 one work week. Then evaluate.

16. The ordered pair (1, 5) belongs to a function f. Explain why the ordered pair (2, 1) cannot belong to f^{-1}.

17. **T** Given $f(x) = k(2 + x)$, find the value of k if $f^{-1}(-2) = -3$.

18. **C** Use a chart like the one shown to summarize what you have learned about the inverse of a linear function.



Extending

19. *Self-inverse* functions are their own inverses. Find three linear functions that are self-inverse.

20. Determine the inverse of the inverse of $f(x) = 3x + 4$.