1.4 Determining the Domain and Range of a Function

GOAL

Use tables, graphs, and equations to find domains and ranges of functions.

YOU WILL NEED

- graph paper
- graphing calculator

LEARN ABOUT the Math

The CN Tower in Toronto has a lookout level that is 346 m above the ground.

A gull landing on the guardrail causes a pebble to fall off the edge.

The speed of the pebble as it falls to the ground is a **function** of how far it has fallen. The equation for this function is $v(d) = \sqrt{2gd}$, where

- d is the distance, in metres, the pebble has fallen
- $v(d)$ is the speed of the pebble, in metres per second (m/s)
- g is the acceleration due to gravity—about 9.8 metres per second squared (m/s^2)

? How can you determine the domain and range of the function $v(d)$?

EXAMPLE 1 Selecting a strategy to determine the domain and range

Determine the domain and range of $v(d)$, the pebble's speed.

Sally's Solution: Using a Graph

The pebble falls a total distance of 346 m. So the domain is $0 \leq d \leq 346$.

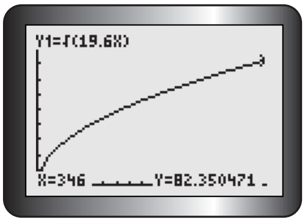
The distance d is 0 m when the pebble first falls off the edge and 346 m when the pebble lands on the ground. So d can take only values that lie in between 0 and 346. This gave me the domain for the function.

$2 \times 9.8 = 19.6$, so the function is

$v(d) = \sqrt{19.6d}$ for $0 \leq d \leq 346$

I entered Y= $\sqrt{19.6x}$ into my graphing calculator. I used $0 \leq X \leq 346$, Xscl 20 and $0 \leq Y \leq 100$, Yscl 10 for WINDOW settings.

Range:



I saw that the graph started at the origin.

The pebble starts with no velocity. So 0 is the minimum value of the range.

The graph showed me that as the pebble's distance increases, so does its velocity. The pebble must be travelling the fastest when it hits the ground. This happens when $d = 346$. The maximum value of the range is $v(346)$. I evaluated this using the value operation.

$v(346) \doteq 82.4$

So the range is $0 \leq v(d) \leq 82.4$.

Domain $= \{d \in \mathbf{R} \mid 0 \leq h \leq 346\}$

Range $= \{v(d) \in \mathbf{R} \mid 0 \leq v(d) \leq 82.4\}$

I used set notation to write the domain and range. I defined them as sets of **real numbers.**

real numbers
numbers that are either rational or irrational; these include positive and negative integers, zero, fractions, and irrational numbers such as $\sqrt{2}$ and π

Communication *Tip*

Set notation can be used to describe domains and ranges. For example, $\{x \in \mathbf{R} \mid 0 \leq x < 50\}$ is read "the set of all values x that belong to the set of real numbers, such that x is greater than or equal to 0 and less than 50." The symbol "|" stands for "such that."

David's Solution: Using the Function Equation

$d = 0$ when the pebble begins to fall, and $d = 346$ when it lands.

So the domain is $0 \leq d \leq 346$.

I found the domain by thinking about all the values that d could have. d is 0 m when the pebble first falls off the edge and 346 m when the pebble lands on the ground. So d must take values between 0 and 346.

The pebble starts with speed 0 m/s.

The pebble *fell* off the edge, so the speed was zero at the start.

$v(0) = \sqrt{19.6(0)} = 0$

I used the equation as a check.

As the pebble falls, its speed increases.

I knew that the pebble would gain speed until it hit the ground.

When the pebble lands, $d = 346$.

$v(346) = \sqrt{19.6(346)}$

$= 82.4$, to one decimal place

I used the function equation to find how fast the pebble was falling when it landed.

The domain is $\{d \in \mathbf{R} \mid 0 \leq d \leq 346\}$ and the range is $\{v(d) \in \mathbf{R} \mid 0 \leq v(d) \leq 82.4\}$.

I used set notation to write the domain and range.

Reflecting

A. Why did Sally need to think about the possible values for distance fallen before she graphed the function?

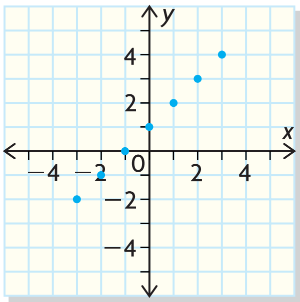
B. What properties of the square root function helped David use the given equation to find the domain and range?

APPLY the Math

EXAMPLE 2 Determining domain and range from graphs

For each relation, state the domain and range and whether the relation is a function.

a)



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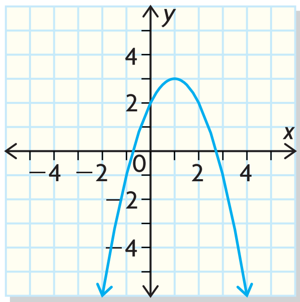

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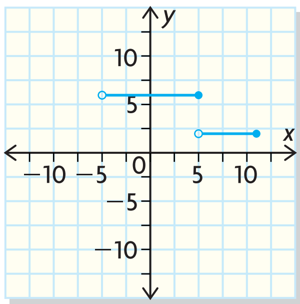

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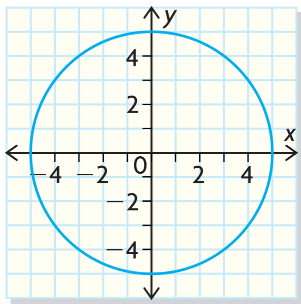


Melanie's Solution

a) Domain $= \{x \in \mathbf{I} \mid -3 \le x \le 3\}$, or $\{-3, -2, -1, 0, 1, 2, 3\}$

Range $= \{y \in \mathbf{I} \mid -2 \le y \le 4\}$, or $\{-2, -1, 0, 1, 2, 3, 4\}$

I noticed that the x-coordinates were all the integers from -3 to 3 and the y-coordinates were all the integers from -2 to 4.

The graph is that of a function.

The graph passes the vertical-line test.

b) Domain $= \{x \in \mathbf{R} \mid -5 < x \leq 11\}$

An open circle on the graph shows that the endpoint of the line is not included in the graph. A closed circle means that the endpoint is included. So, x cannot be -5, but it can be 11.

Range $= \{2, 6\}$

There are only two y-values.

This is a function.

The graph passes the vertical-line test.

c) Domain $= \{x \in \mathbf{R}\}$

Range $= \{y \in \mathbf{R} \mid y \leq 3)\}$

The graph is a parabola with a maximum value at the vertex, which is the point (1, 3).

Therefore, x can be any real number, but y cannot be greater than 3.

This is a function.

The graph passes the vertical-line test.

d) Domain $= \{x \in \mathbf{R} \mid -5 \leq x \leq 5\}$

Range $= \{y \in \mathbf{R} \mid -5 \leq y \leq 5\}$

This is not a function.

The graph is a circle with centre (0, 0) and radius of 5. The graph fails the vertical-line test. There are many vertical lines that cross the graph in two places

EXAMPLE 3 Determining domain and range from the function equation

Determine the domain and range of each function.

a) $f(x) = 2x - 3$ **b)** $g(x) = -3(x + 1)^2 + 6$ **c)** $h(x) = \sqrt{2 - x}$

Jeff's Solution

a) $f(x) = 2x - 3$

This is the equation of a straight line that goes on forever in both directions.

This is a linear function, so x and y can be any value.

Domain $= \{x \in \mathbf{R}\}$

Range $= \{y \in \mathbf{R}\}$

x and $f(x)$ can be any numbers.

I used y instead of $f(x)$ to describe the range.

b) $g(x) = -3(x + 1)^2 + 6$

This is the equation of a parabola that opens down, so y can never be more than its value at the vertex.

This is a quadratic equation in vertex form. The function has a maximum value at the vertex $(-1, 6)$. x can be any value.

Any value of x will work in the equation, so x can be any number.

Domain $= \{x \in \mathbf{R}\}$

Range $= \{y \in \mathbf{R} \mid y \leq 6\}$

c) $h(x) = \sqrt{2 - x}$

$2 - x \geq 0$

$2 - x \geq 0$ as long as $x \leq 2$

Domain $= \{x \in \mathbf{R} \mid x \leq 2\}$

$\sqrt{2 - x}$ means the positive square root, so y is never negative.

Range $= \{y \in \mathbf{R} \mid y \geq 0\}$

You cannot take the square root of a negative number, so $2 - x$ must be positive or zero.

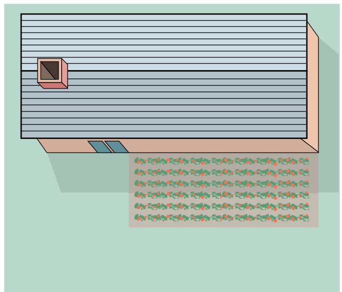
I thought about different possible values of x. 2 is okay, since $2 - 2 = 0$, but 4 is not, since $2 - 4$ is negative. I realized I had to use values less than or equal to 2.

EXAMPLE 4 Determining domain and range of an area function

Vitaly and Sherry have 24 m of fencing to enclose a rectangular garden at the back of their house.

a) Express the area of the garden as a function of its width.

b) Determine the domain and range of the area function.

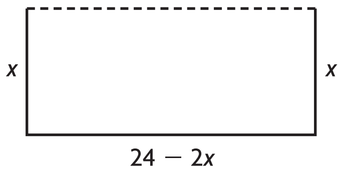


Jenny's Solution

a) Let the width of the garden be x m. Then the length is $(24 - 2x)$ m.

They need fencing on only three sides of the garden because the house forms the last side.

To find the length, I subtracted the two widths from 24.



Let the area be $A(x)$.

$$A(x) = x(24 - 2x)$$
$$A(x) = -2x(x - 12)$$

Area = width × length

I factored out -2 from $24 - 2x$ to write the function in factored form.

b) The smallest the width can approach is 0 m. The largest the width can approach is 12 m.

Domain $= \{x \in \mathbf{R} \mid 0 < x < 12\}$

This is a quadratic function that opens down. It has two zeros, at 0 and 12. The vertex lies halfway in between the zeros, above the x-axis, so the numbers in the domain have to be between 0 and 12. Any number ≤ 0 or ≥ 12 will result in a zero or negative area, which doesn't make sense.

$$x = (0 + 12) \div 2$$
$$x = 6$$

The vertex is halfway between $x = 0$ and $x = 12$. The x-coordinate of the vertex is 6.

$$A(6) = -2(6)(6 - 12)$$
$$= 72$$

The area ranges from 0 to 72 m^2.

Range $= \{A(x) \in \mathbf{R} \mid 0 < A(x) \leq 72\}$

I substituted $x = 6$ into the area function to find the y-coordinate of the vertex. Since area must be a positive quantity, all the output values of the function must lie between 0 and 72.

In Summary

Key Ideas

- The domain of a function is the set of values of the independent variable for which the function is defined. The range of a function depends on the equation of the function. The graph depends on the domain and range.
- The domain and range of a function can be determined from its graph, from a table of values, or from the function equation. They are usually easier to determine from a graph or a table of values.

Need to Know

- All linear functions include all the real numbers in their domains. Linear functions of the form $f(x) = mx + b$, where $m \neq 0$, have range $\{y \in \mathbf{R}\}$. Constant functions $f(x) = b$ have range $\{b\}$.
- All quadratic functions have domain $\{x \in \mathbf{R}\}$. The range of a quadratic function depends on the maximum or minimum value and the direction of opening.
- The domains of square root functions are restricted because the square root of a negative number is not a real number. The ranges are restricted because the square root sign refers to the positive square root. For example,
  - The function $f(x) = \sqrt{x}$ has domain $= \{x \in \mathbf{R} \mid x \geq 0\}$ and range $= \{y \in \mathbf{R} \mid y \geq 0\}$.
  - The function $g(x) = \sqrt{x - 3}$ has domain $= \{x \in \mathbf{R} \mid x \geq 3\}$ and range $= \{y \in \mathbf{R} \mid y \geq 0\}$.
- When working with functions that model real-world situations, consider whether there are any restrictions on the variables. For example, negative values often have no meaning in a real context, so the domain or range must be restricted to nonnegative values.

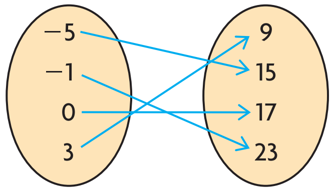
CHECK Your Understanding

1. State the domain and range of each relation.

a)

Year of Birth	Life Expectancy (years)
1900	47.3
1920	54.1
1940	62.9
1960	69.7
1980	73.7
2000	77.0

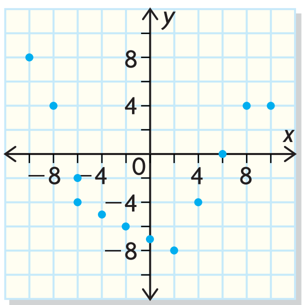
b)



c) $\{(-4, 7), (0, 5), (0, 3), (3, 0), (5, -1)\}$

2. State the domain and range of each relation.

a)



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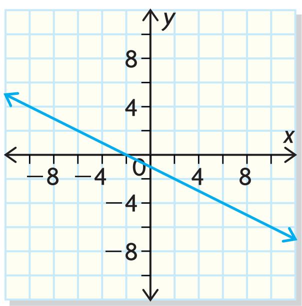

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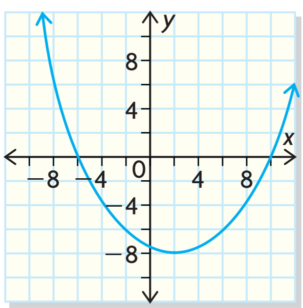

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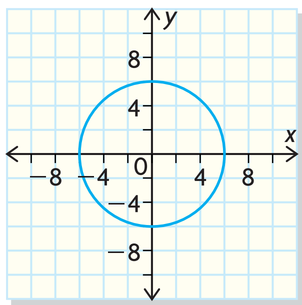

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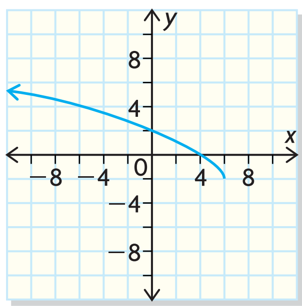

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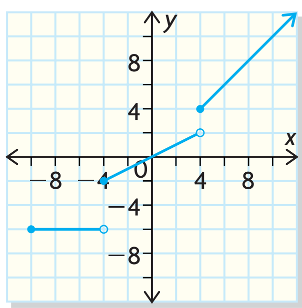
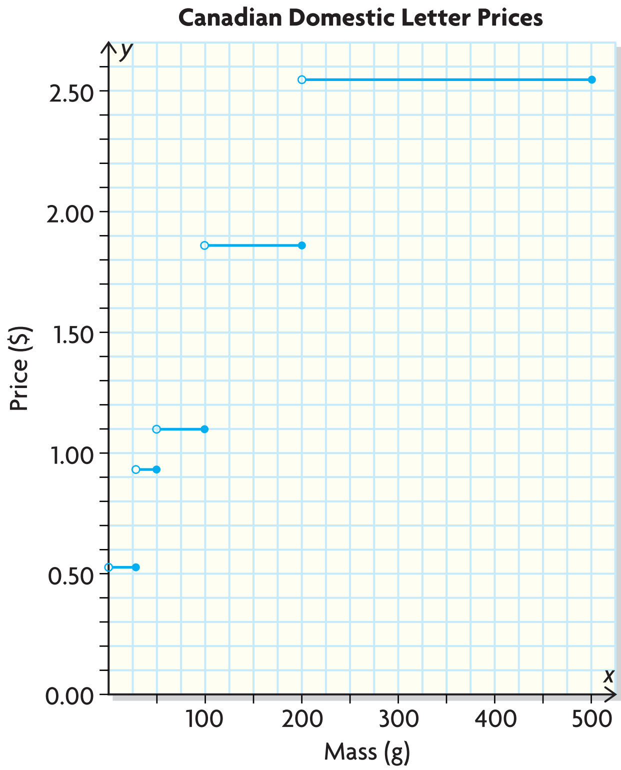


3. Identify which of the relations in questions 1 and 2 are functions.

4. Determine the domain and range of the function $f(x) = 2(x - 1)^2 - 3$ by sketching its graph.

PRACTISING

5. The graph shows how 2007 prices for mailing letters in Canada vary with mass.



a) Explain why this relation is a function. Why is it important for this to be so?

b) State the domain and range of the function.

6. The route for a marathon is 15 km long. Participants may walk, jog, run, or cycle. Copy and complete the table to show times for completing the marathon at different speeds.

Speed (km/h)	1	2	3	4	5	6	8	10	15	20
Time (h)	15.0	7.5								

Graph the relation in the table and explain how you know that it is a function. State the domain and range of the function.

7. A relation is defined by $x^2 + y^2 = 36$.

K **a)** Graph the relation.

b) State the domain and range of the relation.

c) Is the relation a function? Explain.

8. Write a function to describe coffee dripping into a 10-cup carafe at a rate of 1 mL/s. State the domain and range of the function (1 cup = 250 mL).

9. Determine the domain and range of each function.

a) $f(x) = -3x + 8$

b) $g(x) = -0.5(x + 3)^2 + 4$

c) $h(x) = \sqrt{x - 1}$

d) $p(x) = \frac{2}{3}(x - 2)^2 - 5$

e) $q(x) = 11 - \frac{5}{2}x$

f) $r(x) = \sqrt{5 - x}$

10. A ball is thrown upward from the roof of a 25 m building. The ball reaches a
A height of 45 m above the ground after 2 s and hits the ground 5 s after being thrown.

a) Sketch a graph that shows the height of the ball as a function of time.

b) State the domain and range of the function.

c) Determine an equation for the function.

11. Write the domain and range of each function in set notation.

a) $f(x) = 4x + 1$

b) $f(x) = \sqrt{x - 2}$

c) $f(x) = 3(x + 1)^2 - 4$

d) $f(x) = -2x^2 - 5$

12. Use a graphing calculator to graph each function and determine the domain and range.

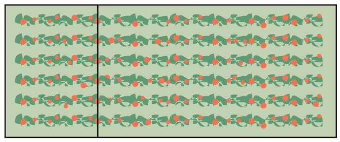
a) $f(x) = \sqrt{3 - x} + 2$

b) $g(x) = x^2 - 3x$

c) $h(x) = \frac{1}{x^2}$

d) $p(x) = \sqrt{x^2 - 5}$

13. A farmer has 450 m of fencing to enclose a rectangular area and divide it into
T two sections as shown.



a) Write an equation to express the total area enclosed as a function of the width.

b) Determine the domain and range of this area function.

c) Determine the dimensions that give the maximum area.

14. Determine the range of each function if the domain is $\{-3, -1, 0, 2.5, 6\}$.

a) $f(x) = 4 - 3x$

b) $f(x) = 2x^2 - 3x + 1$

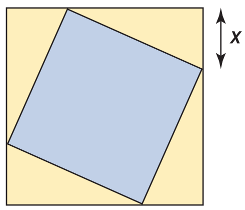
15. Explain the terms "domain" and "range" as they apply to relations and
C functions. Describe, with examples, how the domain and range are determined from a table of values, a graph, and an equation.

Extending

16. **a)** Sketch the graph of a function whose domain is $\{x \in \mathbf{R}\}$ and range is $\{y \in \mathbf{R} \mid y \leq 2)\}$.

b) Sketch the graph of a relation that is not a function and whose domain is $\{x \in \mathbf{R} \mid x \geq -4\}$ and range is $\{y \in \mathbf{R}\}$.

17. You can draw a square inside another square by placing each vertex of the inner square on one side of the outer square. The large square in the diagram has side length 10 units.



a) Determine the area of the inscribed square as a function of x.

b) Determine the domain and range of this area function.

c) Determine the perimeter of the inscribed square as a function of x.

d) Determine the domain and range of this perimeter function.