Using Points to Determine the Equation of a Line

GOAL

Determine the equation of a line given information about related points.

LEARN ABOUT the Math

Ken's Kanine Kennel provides suites that dogs in the same family can share. Ken's charges a room fee for the family plus an additional amount for each dog. One day's stay costs $71 for 2 dogs and $113 for 5 dogs. Julie wants to know the daily cost to board her 3 dogs.

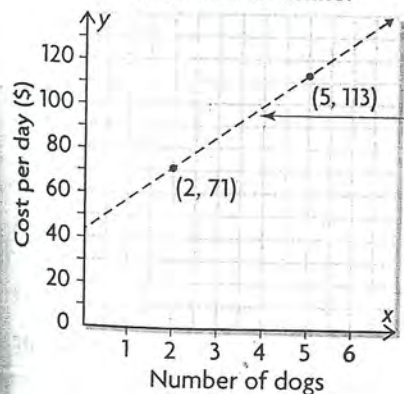**?** How can you determine the equation of this relationship?

EXAMPLE 1   Using a strategy involving the slope formula and equation solving

Determine the equation that describes the relationship between the number of dogs and the daily boarding cost.

Katerina's Solution

Let x represent the number of dogs and y represent the daily cost.

> I chose x to represent the number of dogs because it was the independent variable, and y to represent the cost because the cost depended on the number of dogs.

Cost per Day vs. Number of Dogs at Ken's Kanine Kennel

> I sketched the given information so that I could get an idea of what was happening.

$y = mx + b$

> I knew that the equation of a line can have the form $y = mx + b$. I needed to determine the slope (m) and the y-intercept (b).

Determine the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{113 - 71}{5 - 2}$$

$$= \frac{42}{3}$$

$$= 14$$

I used the slope formula and the coordinates of the two points.

(2, 71)   (5, 113)

x_1 y_1 x_2 y_2

$$y = 14x + b$$

I substituted the value of the slope into my equation.

Determine the y-intercept:

$$71 = 14(2) + b$$
$$71 = 28 + b$$
$$71 - 28 = 28 - 28 + b$$
$$43 = b$$

From my sketch, I could tell the y-intercept was between 40 and 50. I also knew that the point (2, 71) was on the line, so if I substituted 2 for x in the equation, then the y-value had to be 71. That gave me enough information to solve for b.

$$y = 14x + 43$$

Since I knew the values for both m and b, I could write the equation.

Check:

Left Side   Right Side

$$14x + 43$$

113    $$= 14(5) + 43$$
$$= 70 + 43$$
$$= 113$$

I checked to see if boarding 5 dogs would cost $113 using my equation. Since both sides were equal, I knew that my equation was correct.

The relationship between the daily boarding cost and the number of dogs is $y = 14x + 43$. This means that Ken's charges a room fee of $43, plus an additional $14 per dog per day.

Reflecting

A. How was Katerina able to determine the slope and the exact value of the y-intercept of the line?

B. Explain why Katerina could have used the point (5, 113) to determine a value for b.

C. Katerina determined the value of m before she determined the value for b. Could she have determined b before m? Explain.

APPLY the Math

EXAMPLE 2 Using the slope and one point to determine the equation of a line

Determine the equation of the line that has a slope of 4 and passes through the point (2, 6).

Sherif's Solution

$$y = mx + b$$
$$y = 4x + b$$

I was given a value for the slope, so I substituted 4 for m.
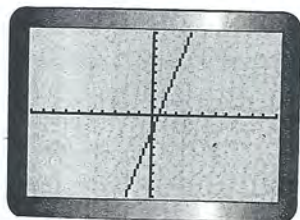
Determine the y-intercept:

$$6 = 4(2) + b$$

(2, 6) was a point on the line, so I substituted its coordinates for x and y.

$$6 = 8 + b$$
$$6 - 8 = 8 - 8 + b$$
$$-2 = b$$

The equation of the line is $y = 4x - 2$.

I solved the equation for b.

I graphed my equation and used the table to verify that my line passed through the point (2, 6).

EXAMPLE 3 Reasoning from properties of a line to determine its equation

Determine the equation of the line that has a slope of $-\dfrac{1}{3}$ and the same y-intercept as the line $2x - 4y + 7 = 0$.

Anayis's Solution

Determine the slope:

$$y = mx + b$$
$$y = -\frac{1}{3}x + b$$

I substituted the given value of $-\dfrac{1}{3}$ for the slope, m.

Determine the _y_-intercept:

$$2x - 4y + 7 = 0$$

> I needed to determine the _y_-intercept of $2x - 4y + 7 = 0$ because the lines had the same _y_-intercept.

$$2(0) - 4y + 7 = 0$$

> I knew that the _x_-coordinate was 0 for the point where the line crossed the _y_-axis.

$$-4y + 7 = 0$$
$$-4y + 7 - 7 = 0 - 7$$
$$-4y = -7$$
$$\frac{-4y}{-4} = \frac{-7}{-4}$$
$$y = \frac{7}{4}$$

> I used inverse operations to solve the equation for _y_.

$$b = \frac{7}{4}$$

> I knew that the b in the equation represented the _y_-intercept.
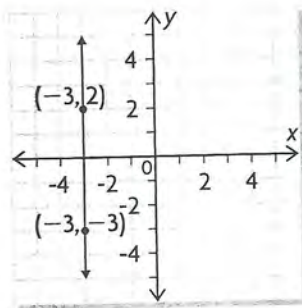
The equation of the line is

$$y = -\frac{1}{3}x + \frac{7}{4}.$$

EXAMPLE 4   Using reasoning to determine the equation of a line through two points with the same _x_-coordinate

Determine the equation of the line that passes through the points $(-3, -3)$ and $(-3, 2)$.

Omar's Solution

This line is vertical. The equation of the line is $x = -3$.

> From the sketch of the graph, I could see that the line was vertical and that every point on the line had −3 for its _x_-coordinate.

EXAMPLE 5   Using the equation of a line to solve a problem

The speed of sound in air can be calculated based on temperature using a linear relation. At 10 °C the speed of sound is 337.4 m/s, and at 21.5 °C the speed is 344.3 m/s. What is the speed of sound when the temperature is 32.3 °C?

Suhaila's Solution

Let T represent the temperature and S represent the speed of sound.

> I chose variables to represent each unknown quantity.

$$S = mT + b$$

> I wanted to find an equation for the relationship first. Using $y = mx + b$, I replaced y with S since sound is the dependent variable and x with T since temperature is the independent variable.

Determine the slope:

$$m = \frac{344.3 - 337.4}{21.5 - 10}$$

$$= \frac{6.9}{11.5}$$

$$= 0.6$$

> I substituted the points $(10, 337.4)$ and $(21.5, 344.3)$ into the slope formula.

This means the speed increases 0.6 m/s for each Celsius degree increase in temperature.

Determine the S-intercept:

$$S = 0.6T + b$$

> I substituted the slope value 0.6 into my equation.

$$337.4 = 0.6(10) + b$$
$$337.4 = 6 + b$$
$$337.4 - 6 = 6 - 6 + b$$
$$331.4 = b$$

> I used the point $(T, S) = (10, 337.4)$ in my equation to solve for b.

This means that when the temperature is 0 °C the speed of sound is 331.4 m/s. $S = 0.6T + 331.4$ represents the speed of sound S at temperature T.

$$S = 0.6(32.3) + 331.4$$
$$S \doteq 350.8$$

I substituted the given temperature 32.3 °C into my equation to solve the equation.

At 32.3 °C, the speed of sound is 350.8 m/s.

In Summary

Key Ideas

- You can determine the equation of a line in the form $y = mx + b$ if you know two points on the line or one point and the slope.

Need to Know

- You can determine the equation of a line as follows:
  - If the slope is not given, and you know two points on the line, use the coordinates of the points to calculate the slope.
  - Substitute the value of the slope for m and the coordinates (x, y) of a point on the line into $y = mx + b$ and solve for b.
  - Use the values of m and b to write the equation of the line.

CHECK *Your Understanding*

1. Complete the table on the right by determining the missing values.

	Slope	y-intercept	Equation
a)	3	5	
b)			$y = 5x + 1$

2. Match each equation to its corresponding graph.
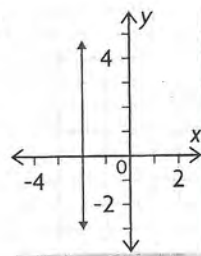
a) $x = -2$

b) $x - 2y = 4$
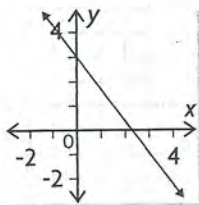
c) $y = -2$

d) $y = -\dfrac{4}{3}x + 3$
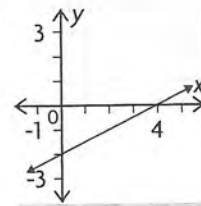
i)
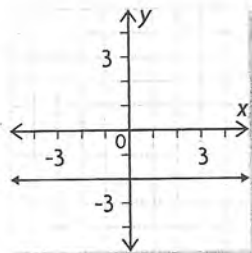
iii)

ii)

iv)

3. Determine the equation of the line with the following characteristics.

   a) has a slope of -2 and passes through the point $A(5, 2)$

   b) passes through the points $B(4, 6)$ and $C(1, -3)$

PRACTISING

4. Complete the table shown below by determining the missing values.

K

	Slope	y-intercept	Equation
a)	-5	3	
b)			$y = \dfrac{4}{3}x - 2$
c)	0	2	
d)			$y = \dfrac{1}{2}x$

5. Determine the equation of each line shown below.

a)

d)

b)

e)

c)

f)
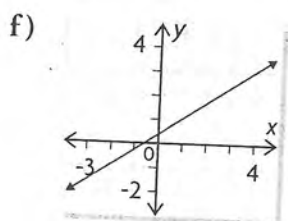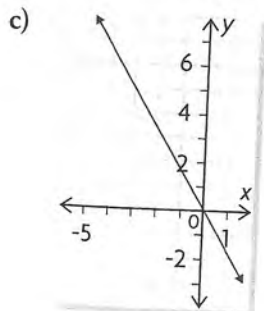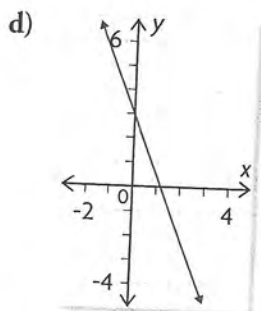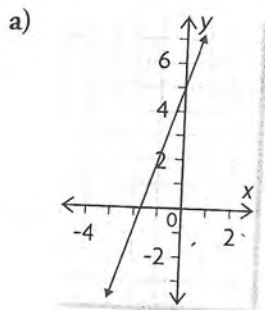
6. If the equation $y = 3x + b$ represents a line that passes through the given point, determine the value of the y-intercept, b.
   a) $(4, 1)$ b) $(-3, 2)$ c) $(1, -3)$

7. If the equation $y = mx + 3$ represents a line that passes through the given point, determine the slope value, m.
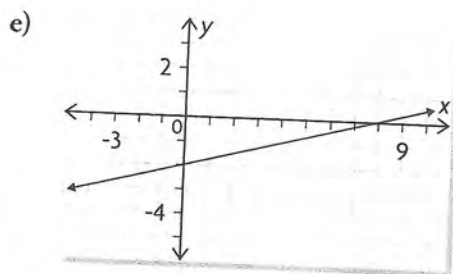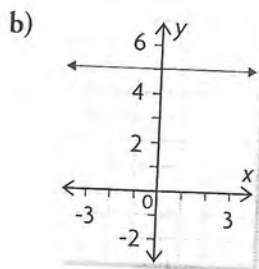   a) $(2, 4)$ b) $(-3, 7)$ c) $(8, 2)$

8. Determine the equation of each line described below.
   a) passing through the point A$(0, 4)$, with a slope of $-\dfrac{8}{9}$

   b) passing through the point A$(3, -5)$, with a slope of $\dfrac{1}{5}$

   c) has an x-intercept of 4 and a y-intercept of -3
   d) has an x-intercept of 6 and passes through the point $(6, 4)$

9. Determine the equation of the line passing through each pair of points.
   a) A$(1, 9)$ and B$(1, -7)$ d) G$(6, 18)$ and H$(-12, 3)$
   b) C$(-8, -3)$ and D$(8, 27)$ e) I$(0, 5)$ and J$(0, 12)$
   c) E$(-12, 7)$ and F$(4, 7)$ f) K$(-5, -1)$ and L$(15, 1)$

10. Determine the equation of the line that has the same x-intercept as the line described by $x - 5y + 10 = 0$, and the same y-intercept as the line $3x + 2y - 6 = 0$.

11. The LeBlanc family is driving home. The LeBlancs are using cruise control so their speed is constant. After 3 h, they are 350 km from home. After 5 h, they are 130 km from home.
    a) Write an equation to represent this distance–time relationship.
    b) What do the slope and y-intercept of your equation mean in this situation?

12. A stress test evaluates the health of a patient's heart. While riding on a stationary bike or running on a treadmill, a patient has his or her heart rate measured by a technician and compared with a safe maximum heart rate. This safe heart rate is based on the patient's age as shown in the graph.

    Maximum Heart Rate vs. Age
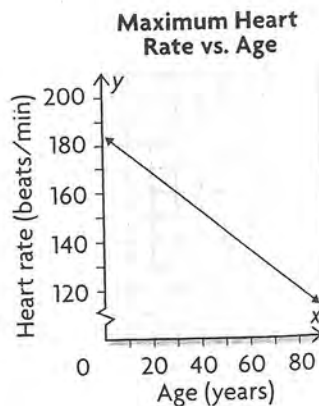
    a) What does the y-intercept represent in this situation?
    b) What does the slope of the graph represent?
    c) Determine the equation for the line.
    d) Ellen is 14 years old. Using your equation, determine her maximum safe heart rate.

13. The local fall fair charges a flat fee for admission plus an additional cost
T for ride tickets. Last year, Kelsey purchased 15 tickets and spent a total
of $19.50. His brother Quinn purchased 36 tickets and spent a total of
$30.00 at the fair.
   a) Determine an equation to represent the relationship between the
   total amount of money spent and the number of tickets purchased.
   b) A ride pass, which gives a person entrance to the park and
   unlimited use of the rides, costs $21. Write the equation for the
   relationship between the total amount spent on a ride pass and the
   number of rides it can be used for.
   c) Last year, Erin used 25 tickets at the fall fair. Should Erin purchase
   tickets again this year, or buy a ride pass? Explain.
   d) Heather only likes the fun house, which requires one ticket. She
   went on this ride 10 times last year. How much money would
   Heather save by purchasing tickets instead of a ride pass?

14. Lori downloads music from the Music Genie site, which charges a
A monthly membership fee plus an amount for each song downloaded.
A three-month record of her activity on the site is shown.

Month	Number of Songs Downloaded	Monthly Bill ($)
January	54	26.90
February	38	25.30
March	21	23.60

   a) Use two points from the table to determine the equation of the
   relationship between numbers of downloads and her monthly bill.
   b) Verify that the third point from the table also satisfies your
   equation.
   c) Lori's brother thinks she should change to Web Waves, which
   doesn't have a membership fee and charges $0.95 per song. Based
   on your calculations, do you think Lori should change music
   companies? Explain.
   d) Digital Beats charges $25 for a monthly membership, with
   unlimited free downloads. Would you recommend Lori change to
   Digital Beats? Explain.

15. Shawn says he can only figure out the equation of a line if he is given
C the y-intercept and the slope of the line. Barb says that she can figure
out the equation using the coordinates of any two points on the line.
With whom do you agree? Why?

Extending

16. Determine the equation of the **median** from Q to the midpoint of PR, in triangle PQR, with P(4, 8), Q(3, 2), and R(6, 4).
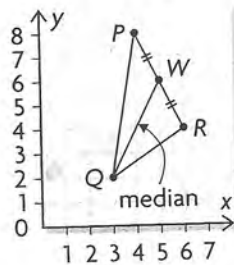
17. Given any two points on a line, the equation of the line can be determined from the point-slope form of the equation of the line:

$$y = m(x - p) + q.$$
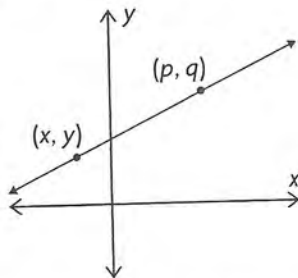
a) Show how the formula can be developed using the slope formula,

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

b) Use the point-slope form of the equation of a line to determine the equation of a line that has a slope of 3 and passes through the point (1, 2).

       p q

c) Determine the equation of the line in part b) using $y = mx + b$ to verify that the new formula works.

d) Use the point-slope form of the equation of a line to determine the equation for each of the following lines.
   i) passing through the points (4, −6) and (5, −1)
   ii) passing through the points (3, −1) and (9, 3)
   iii) passing through the points (4, 5) and (3, 9)

9. a) $-\dfrac{1}{2}$ **c)** -1 **e)** 0

b) $\dfrac{11}{10}$ **d)** $\dfrac{5}{3}$ **f)** undefined

10. a) no   **b)** no   **c)** yes   **d)** no

11. a) $k = -\dfrac{18}{5}$ or -3.6 **c)** $k = 4$

b) $k = 1$ **d)** $k = \dfrac{38}{5}$ or 7.6

12. $15 per person

13. 89 km/h

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1. a) $y = 3x + 5$ **b)** 5; 1

2. a) iii   **b)** iv   **c)** i   **d)** ii

3. a) $y = -2x + 12$ **b)** $y = 3x - 6$

4. a) $y = -5x + 3$ **b)** $\dfrac{4}{3}$; -2 **c)** $y = 2$ **d)** $\dfrac{1}{2}$; 0

5. a) $y = 3x + 5$ **c)** $y = -\dfrac{5}{3}x$ **e)** $y = \dfrac{1}{4}x - 2$

b) $y = 5$ **d)** $y = -\dfrac{5}{2}x + 3$ **f)** $y = \dfrac{2}{3}x + \dfrac{1}{3}$

6. a) -11 **b)** 11   **c)** -6

7. a) $\dfrac{1}{2}$ **b)** $\dfrac{-4}{3}$ **c)** $\dfrac{-1}{8}$

8. a) $y = \dfrac{-8}{9}x + 4$ **c)** $y = \dfrac{3}{4}x - 3$

b) $y = \dfrac{1}{5}x - \dfrac{28}{5}$ **d)** $x = 6$

9. a) $x = 1$ **c)** $y = 7$ **e)** $x = 0$

b) $y = \dfrac{15}{8}x + 12$ **d)** $y = \dfrac{5}{6}x + 13$ **f)** $y = \dfrac{1}{10}x - \dfrac{1}{2}$

10. $y = \dfrac{3}{10}x + 3$

11. a) $y = -110x + 680$, where x is the time in hours and y is the distance from home in kilometres.

b) The slope of -110 means that the distance is decreasing at a rate of 110 km/h. The y-intercept of 680 means they began their trip 680 km away from home.

12. a) maximum heart rate in a stress test for a newborn baby

b) the rate of decline of maximum heart rate in a stress test over the years

c) $y = \dfrac{-4}{5}x + 184$

d) about 173 beats/min

13. a) $C = 0.50t + 12$, where C is the total cost, in dollars, and t is the number of ride tickets purchased.

b) $C = 21$, where C is the total cost, in dollars.

c) Answers may vary, e.g., since 25 tickets cost $24.50, she should buy the ride pass.

d) $4

14. a) $B = 0.1d + 21.50$, where B is the total bill and d is the number of songs downloaded.

b) $B = 0.1(21) + 21.50 = 23.60, so the third point also satisfies the equation.

c) Answers may vary, e.g., if she downloads more than 25 songs per month on average she should not change companies.

d) Answers may vary, e.g., if she averages 35 downloads or more per month then she should consider changing to Digital Beats.

15. Barb is correct. The slope and y-intercept can be determined from any two pieces of information, such as two points, or the x- and y-intercepts.

16. $y = 2x - 4$

17. a) Substitute the coordinates into the slope formula $\dfrac{y - q}{x - p} = m$ then isolate y.

b) $y = 3(x - 1) + 2$, which simplifies to $y = 3x - 1$

c) $y = 3x - 1$

d) i) $y = 5(x - 4) - 6$ or $y = 5(x - 5) - 1$; both simplify to $y = 5x - 26$

ii) $y = \dfrac{2}{3}(x - 3) - 1$ or $y = \dfrac{2}{3}(x - 9) + 3$; both simplify to $y = \dfrac{2}{3}x - 3$

iii) $y = -4(x - 4) + 5$ or $y = -4(x - 3) + 9$; both simplify to $y = -4x + 21$

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1. a) Answers may vary, e.g., $y = -\dfrac{3}{2}x - 9$.

b) Answers may vary, e.g., $y = \dfrac{2}{3}x + 9$.

2. a, g, and f are parallel; perpendicular pairs: b and e, c and h

3. a) perpendicular   **c)** neither   **e)** parallel

b) perpendicular   **d)** parallel   **f)** neither

4. PQ is parallel to KL; AB $\perp$ PQ; AB $\perp$ KL; GH $\perp$ UV.

5. Perpendicular; one line is horizontal and one line is vertical, so they are perpendicular lines.

6. a) $y = 4$

b) $y = -8$

c) horizontal line equation has the form: $y = y$-coordinate of point through which it passes

7. a) $x = -9$

b) $x = 6$

c) vertical line equation has the form: $x = x$-coordinate of point through which it passes

8. a) $y = 3x - 14$ **c)** $y = \dfrac{-3}{2}x + 3$

b) $y = \dfrac{-1}{3}x - 4$ **d)** $y = \dfrac{-3}{2}x + 6$

9. $y = \dfrac{-3}{4}x - 3$

10. $y = \dfrac{-5}{2}x + \dfrac{25}{2}$

11. a) yes   **b)** no

12. $m_{AB} = -\dfrac{4}{7}$ $m_{AC} = \dfrac{7}{4}$ $m_{CD} = -\dfrac{4}{7}$ $m_{DB} = \dfrac{7}{4}$

AB is perpendicular to AC and to DB.
CD is perpendicular to AC and to DB.
The quadrilateral ABDC has four right angles, so it must be a rectangle.

13. $m_{EF} = -\dfrac{7}{4}$ $m_{FG} = -\dfrac{1}{6}$ $m_{GH} = -\dfrac{7}{4}$ $m_{HE} = -\dfrac{1}{6}$

EF and GH are parallel.
FG and HE are parallel.
No sides are perpendicular.
The quadrilateral EFGH is a parallelogram but not a rectangle.

14. 795 cm

15. a) -4.5 **b)** 1.5