8.6.4

(a) Graph the lines $3x - 2y = k$ for $k = 1, 2, 3,$ and 4.

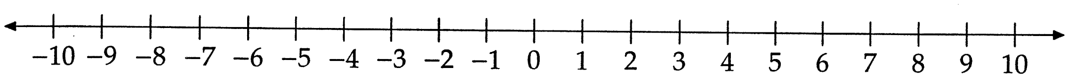
(b) How are the lines related?

(c)★ How can you quickly use the graph of $3x - 6y = 4$ to produce the graph of $3x - 6y = 10$?

8.6.5 Find a and b if the graph of $ax + 2y = b$ passes through $(2, 5)$ and is parallel to $3x - 5y = 7$.

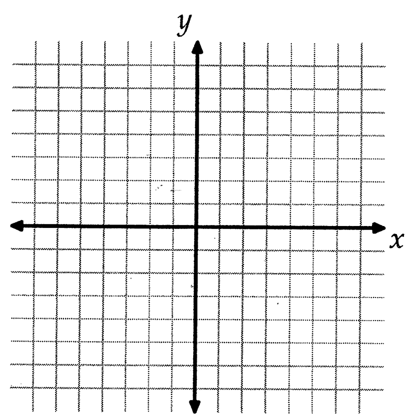
8.7 Summary

The **number line**, shown below, gives us a way to visually represent numbers.



The **absolute value** of a number equals its distance from 0 on the number line. We denote the absolute value of a number x as $|x|$. So, for example, $|-5| = 5$.

We can create visual representations of equations that have two variables using the **Cartesian plane**. The bold horizontal line is called the ***x*-axis** and the bold vertical line is called the ***y*-axis**. The center of the plane, where the axes meet, is called the **origin**.



Each point on the Cartesian plane is represented by an **ordered pair** of numbers, (x, y). These numbers denote the position of the point *relative to the origin*. We call the two numbers in an ordered pair the **coordinates** of the point. By convention, we call the horizontal (left-right) coordinate the ***x*-coordinate** and we call the vertical (up-down) coordinate the ***y*-coordinate**. Points that have integers for both coordinates are called **lattice points**.

Important: The distance in the plane between the points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This is often referred to as the **distance formula**.

Important: The midpoint of the segment connecting the points with coordinates (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

The **graph** of an equation consists of all the points (x, y) on the Cartesian plane that satisfy the equation.

Important: The graph of an equation of the form $Ax + By = C$, where A, B, and C are constants and A and B are not both 0, is a straight line.

Important: The **slope**, m, of the line through the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Important: Just knowing whether the slope of a line is positive, negative, 0, or undefined gives us information about the line.

- Slope is **positive**: The line goes upward as it goes from left to right.
- Slope is **negative**: The line goes downward as it goes from left to right.
- Slope is **0**: The line is horizontal. Its equation is of the form $y = k$, for some constant k.
- Slope is **undefined**: The line is vertical. Its equation is of the form $x = h$, for some constant h.

Important: Slope can be thought of as a measure of steepness. A line with a large positive slope is very "steep," meaning it is close to a vertical line. A line with a very small positive slope close to 0, such as $1/10$, is not at all steep; it's close to a horizontal line.

We can apply this to negative slopes as well. Lines with slopes that are very negative are nearly vertical, while lines with negative slopes very close to 0, such as -0.01, are nearly horizontal.

The ***x*-intercepts** of a graph are the points where the graph meets the x-axis, and the ***y*-intercepts** of a graph are the points where the graph meets the y-axis.

Three commonly used forms of linear equations are:

- **Point-slope form.** The line through the point (x_1, y_1) with slope m is the graph of the equation $y - y_1 = m(x - x_1)$.
- **Standard form.** The standard form of a linear equation is $Ax + By = C$, where $A > 0$ (for non-horizontal lines) and, if possible, A, B, and C are integers with no common divisor. If $B \neq 0$, then the slope of the graph of $Ax + By = C$ is $-A/B$.
- **Slope-intercept form.** The graph of the equation $y = mx + b$ is a line with slope m and y-intercept $(0, b)$.

Important: There are three possibilities for the number of solutions of a system that consists of a pair of two-variable linear equations:

- **No Solutions.** If the graphs of the equations are parallel lines, then the lines never intersect. Consequently, there is no point that is on both lines and therefore no solutions to the system.
- **One Solution.** If the graphs of the equations are not either the same line or two different parallel lines, then they are lines that meet at exactly one point. This point is the only solution to the system of equations.
- **Infinitely Many Solutions.** If the graphs of the two equations produce exactly the same line, then every point on the line is a solution to the system of equations. So, there are infinitely many solutions.

Important: If two lines have the same slope, then they are either parallel (meaning they never intersect) or they are the same line. Conversely, any two parallel lines have the same slope.

Important: **Perpendicular lines** are lines that meet at 90° angles. If the slopes of two lines have a product of -1, then the two lines are perpendicular. Furthermore, if two lines are perpendicular (and neither line is a vertical line), then the product of their slopes must be -1.

roblem Solving Strategies

Concepts:

- The intercepts of a line are easy to find and easy to graph. So, we can often most quickly graph a line given its equation by finding the x- and y-intercepts, plotting these two points, and then drawing the line through them.
- Always be on the lookout for ways to rearrange equations into more convenient forms. For example, when looking for the slope of a line, slope-intercept form makes finding the slope very easy.
- Understanding what the terms slope and intercept mean can lead to quicker, less algebraic solutions.
- Expressing several different quantities in terms of a single variable allows us to easily relate the quantities.

REVIEW PROBLEMS

8.28 Find t such that $(t, 5)$ lies on the line through $(0, 3)$ and $(-8, 0)$.

8.29 Compute $|1 - |2 - |3 - |4 - |5||||$.

8.30 Consider the points $(1, -2)$ and $(-5, 6)$.

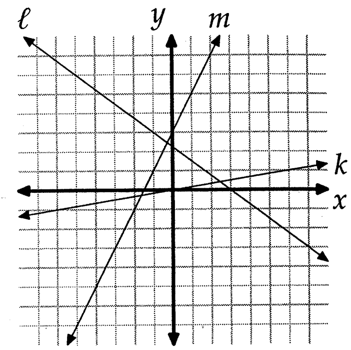
(a) Find the distance between the two points.

(b) What are the coordinates of the midpoint of the segment connecting these points?

(c) Find the slope of the line through both points.

(d) Find the standard form of the equation whose graph is the line through both points.

8.31 A is a constant such that the graph of the equation $Ax - 3y = 6$ passes through the point $(1, -3)$. Find A.



8.32 The lines k, ℓ, and m are graphed at right. One line has slope $-3/4$, one has slope $1/6$, and one has slope 2. Without finding the coordinates of any points on any of the lines, determine which line is which.

8.33 Find the standard form of the equation of line m shown at right.

8.34 Find a formula for the distance from the point (x_1, y_1) to the origin.

8.35 Find the equation, in standard form, of the line that passes through $(4, 0)$ and has slope $-1/3$. Graph the equation.

8.36 What is the slope of the graph of the equation $x = -3$? Graph the equation.

8.37 Find the x-intercept and the y-intercept of the line that passes through $(3, -7)$ and $(-3, 5)$.

8.38 Find the slope, the x-intercept, and the y-intercept of the graph of the equation $2x - 3y + 9 = 0$. Graph the equation.

8.39 Show that for any two real numbers a and b, the number $(a + b)/2$ is the same distance from a as it is from b on the number line.

8.40 Points A, B, C and D are on the Cartesian plane such that the slope of the line through A and B equals the slope of the line through C and D. Are the four points necessarily all on the same line?

8.41 The slope of a line is -2 and its x-intercept is $(5, 0)$. What is the y-intercept of the line?

8.42 Let P be $(-3, 7)$ and Q be $(5, -12)$.

(a) Find the midpoint of $\overline{PQ}$.

(b) Find the point T on $\overline{PQ}$ such that $PT/TQ = 1/3$.

3.43 Find the standard form of the equation of the line through $(4, 1)$ that is perpendicular to the line $2x = -3y + 7$.

3.44 Find the standard form of the equation of the line through $(8, -3)$ that is parallel to the line $3y = 4x + 8$.

3.45 Determine whether each of the following statements is true or false. If it is false, provide an example showing why.

(a) The absolute value of any real number is positive.

(b) The equation of every line can be written in the form $Ax + By = C$ for some real numbers A, B, and C.

(c) The equation of every line can be written in the form $y = mx + b$ for some real numbers m and b.

(d) Every line has both an x-intercept and a y-intercept.

(e) The x-intercept of the line $y = 2x + 3$ is $(0, 3)$.

(f) If the product of the slopes of two lines is -1, then the two lines are perpendicular.

(g) If two lines are perpendicular, then the product of their slopes is -1.

(h) If two lines are parallel, then either they are both vertical or they have the same slope.

.46 A line has equation $x = my + b$ for some real nonzero constants m and b.

(a) In terms of m and/or b, what is the x-intercept of this line?

(b) In terms of m and/or b, what is the slope of this line?

47

(a) Find A if the graph of the equation $Ax + 3y = 5$ is parallel to the graph of $5x - 2y = 4$.

(b) Find B if the graph of the equation $3x = By + 2$ is perpendicular to the graph of $3y = -2x + 4$.

(c) Find A and B if the graph of the equation $Ax + 3y = B$ produces the same line as the graph of the equation $2x + 6y = 7$.

48 Does any line have neither an x-intercept nor a y-intercept? If so, give an example.

Challenge Problems

49 The midpoint of the segment connecting (a, b) and (b, a) is (x, y). Express y in terms of x.

50 If $|x - 3| = 4$, what are all possible real values of x?

51 A line passing through the points $(2a + 4, 3a^2)$ and $(3a + 4, 5a^2)$ has slope $a + 3$, where a is nonzero. nd the value of a.

52 Two lines with nonzero slope and the same y-intercept have the property that the sum of their pes is 0. What is the sum of the x-coordinates of their x-intercepts? **Hints:** 26

8.53 Explain why $\sqrt{x^2} = |x|$ for all real numbers x. **Hints:** 64

8.54 Bob bumped his head and started plotting all his points in reverse order. For example, when he tries to plot $(3, 2)$ he plots $(2, 3)$ instead. A problem in his textbook tells him to graph the line $y = 3x + 2$.

(a) Draw the graph that he will produce, along with the correct line. What are the slopes of these two lines?

(b) Find an equation of the line that he will graph.

(c)★ If he tries to graph a line that has slope $m \neq 0$, what will the slope of the resulting line be, in terms of m? What happens if $m = 0$, or if he tries to graph a vertical line? **Hints:** 118

8.55 Suppose R is $(3, 5)$ and S is $(8, -3)$. Find each point on the line through R and S that is three times as far from R as it is from S. **Hints:** 161

8.56 The bases of two flagpoles are 12 feet apart. One flagpole is 14 feet tall and the other is 11 feet tall. A wire runs directly from the top of the tall flagpole, to the top of the small flagpole, then to the ground, without bending or curving at any point. How far from the base of the small flagpole does the wire reach the ground? **Hints:** 35

8.57★ A line L has a slope of -2 and passes through the point $(r, -3)$. A second line, K, is perpendicular to L at (a, b) and passes through the point $(6, r)$. Find a in terms of r. **Hints:** 222

8.58★ How many points are common to the graphs of the two equations

$$(x - y + 2)(3x + y - 4) = 0 \qquad \text{and} \qquad (x + y - 2)(2x - 5y + 7) = 0?$$

(Source: AHSME) **Hints:** 25

8.59★

(a) Given a real number r between 0 and 1 and two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the plane, find the coordinates in terms of r of the point T on segment PQ such that $PT/PQ = r$.

(b) What happens if you allow r to be greater than 1 in your formula in part (a)? Where on the line connecting P and Q is the resulting T? **Hints:** 28

(c) What happens if you allow r to be negative in your formula in part (a)? Where on the line connecting P and Q is the resulting T? **Hints:** 171

8.60★ What is the smallest possible value of the expression

$$\sqrt{a^2 + 9} + \sqrt{(b - a)^2 + 4} + \sqrt{(8 - b)^2 + 16}$$

for real numbers a and b? **Hints:** 32, 203

Extra! The great mathematician Gottfried Wilhelm Leibniz once noted, *"He who understands Archimedes and Apollonius will admire less the achievements of the foremost men of later times."*

Continued on the next page. . .