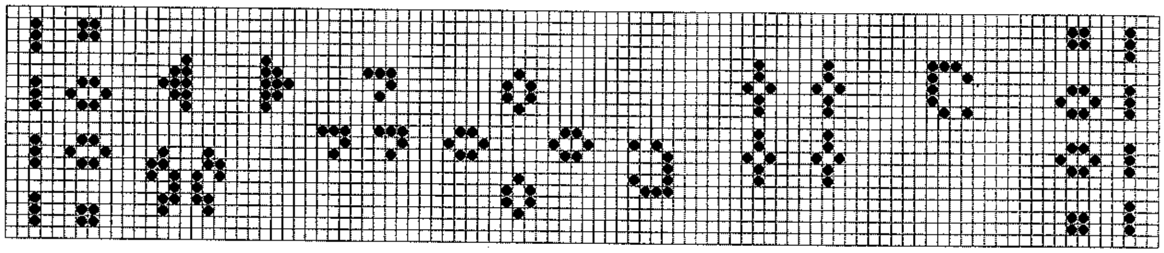


Great things are not done by impulse, but by a series of small things brought together. – Vincent van Gogh

CHAPTER 21 Sequences & Series

A **sequence** is simply a list of numbers, such as

$$2, 4, 6, 8, 10.$$

The sequence above is called a **finite sequence** because it has a finite number of terms. In other words, the sequence ends. Not all sequences end. We use "..." to indicate that a sequence continues forever:

$$2, 4, 6, 8, 10, \ldots$$

Some sequences have obvious patterns, such as

$$1, 2, 3, 4, 5, 6, 7, \ldots \qquad \text{and} \qquad 1, 2, 4, 8, 16, 32, \ldots.$$

In this chapter, we'll study the properties of a couple common examples of sequences that have useful patterns.

When using variables to represent a sequence, we often use the same letter with different subscripts to represent the terms. For example, we might represent a sequence with 5 terms as a_1, a_2, a_3, a_4, a_5.

When we add the terms of a sequence, we form a **series**. Some example series are:

$$1+2+3+4+5+6+7+8$$
$$2+4+6+8+10+12+14$$
$$100+99+98+97+96+95+94$$
$$1+2+4+8+16+32+64+128$$

We could evaluate these series by simply performing lots and lots of addition. However, for some special types of series, there are much simpler ways to compute the sum.

21.1 Arithmetic Sequences

You probably recognize all of the sequences below, and have no trouble guessing what the next few terms are in each:

$$1, 2, 3, 4, 5, 6, 7, \ldots$$
$$2, 4, 6, 8, 10, 12, 14, \ldots$$
$$100, 99, 98, 97, 96, 95, \ldots$$

Each of these is an **arithmetic sequence**. In an arithmetic sequence, the difference between two consecutive terms is always the same. Such a regular pattern in the sequence makes arithmetic sequences relatively easy to understand and analyze.

Problems

Problem 21.1: Consider the sequence

$$-9, -5, -1, 3, 7, \ldots$$

in which each term is 4 greater than the previous term.

(a) Find the sixth, seventh, and eighth terms of the sequence.

(b) Find a formula for the n^{th} term of the sequence.

Problem 21.2:

(a) Must the fifth term of an arithmetic sequence with at least eight terms always be the average of the second and eighth terms?

(b) Suppose that x, y, and z are in an arithmetic sequence such that y is exactly between x and z in the sequence. (In other words, there are just as many terms between x and y as there are between y and z.) Must y be the average of x and z?

Problem 21.3: The sequence

$$4, x_1, x_2, x_3, x_4, 18$$

is an arithmetic sequence. In this problem, we find x_3.

(a) How many "steps" must we take to get from the first term, 4, to the last term, 18?

(b) Use your answer to (a) to determine how large each such step is, and use this to determine x_3.

Problem 21.4: The sum of the second term and the ninth term of an arithmetic sequence is -4. The sum of the third and fourth terms of the same sequence is 4. In this problem, we find the first term of the sequence.

(a) Let the first term be a and the common difference between terms be d, so that the second term is $a + d$. In terms of a and d, what are the third, fourth, and ninth terms?

(b) Use your answers to part (a) to solve the problem.

The pattern in an arithmetic sequence is so simple that we can easily find a formula for all the numbers in the sequence.

Problem 21.1: Consider the sequence

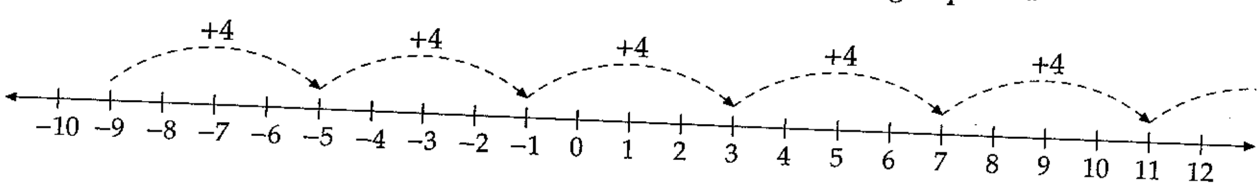
$$-9, -5, -1, 3, 7, \ldots,$$

in which each term is 4 larger than the previous term. Find a formula for the n^{th} term in the sequence.

Solution for Problem 21.1: The pattern in the sequence is clear, and we can easily generate as many more terms as we like:

$$-9, -5, -1, 3, 7, 11, 15, 19, 23, 27, 31, \ldots$$

However, it would be much easier to generate later terms in the sequence if we simply had a formula rather than having to generate a huge list. We can find these later terms by considering how many steps they are from the first term. For example, to get to the 8^{th} term, we start from the first term and take 7 steps. The diagram below illustrates our forming the sequence by taking steps of 4.



Rather than adding 4 seven times, we simply note that taking 7 steps of size 4 means adding $7 \times 4 = 28$ to our first term, so the 8^{th} term is $-9 + 28 = 19$.

Similarly, to get the n^{th} term, we start from the first term and take $n - 1$ rightward steps of size 4 steps. (Make sure you see why it is $n - 1$ steps, not n steps.) Since taking $n - 1$ rightward steps of size 4 means adding $4(n - 1)$ to the first term, this takes us to the number

$$-9 + 4(n - 1).$$

This gives us our formula for the **general term** of the sequence. We can plug in $n = 1, 2, 3, 4$, etc., to produce the sequence. Check it and see! □

We call the size of the "steps" between terms in an arithmetic sequence the **common difference** of the sequence. In exactly the same manner as in the previous problem, we can generate a formula for the general term of any arithmetic sequence given its first term and its common difference.

Important: The n^{th} term of an arithmetic sequence that has first term a and common difference d is

$$a + (n - 1)d.$$

We can use this formula to solve nearly any arithmetic sequence problem. However, a simple understanding of arithmetic sequences usually leads to an even faster solution.

Let's take a look at one property of arithmetic sequences that often simplifies arithmetic sequence problems.

Problem 21.2: Suppose that x, y, and z are in an arithmetic sequence such that y is directly between x and z in the sequence. (In other words, there are just as many terms between x and y as there are between y and z.) Must y be the average of x and z?

Solution for Problem 21.2: To get a feel for the problem, we try a specific case.

Concept: Experimenting with special cases is a great way to develop an understanding of a general statement.

Suppose x is the 2^{nd} term, y is the 5^{th} term, and z is the 8^{th} term, so that y is directly between x and z in the sequence.

Intuitively, it is clear that y is the average of x and z, since x is three steps before y and z is three steps after y. We can prove that y is the average of x and z by letting d be our common difference and noting that

$$x = y - 3d,$$
$$z = y + 3d.$$

Adding these equations gives $x + z = 2y$, so $(x + z)/2 = y$, as desired.

This example gives us a clear path to proving that if y is directly between x and z in an arithmetic sequence, then y is the average of x and z. Let d be the common difference and k be the number of steps between x and y. The number of steps between y and z therefore is also k, so we have

$$x = y - kd,$$
$$z = y + kd.$$

Adding these equations gives $x + z = 2y$, so $(x + z)/2 = y$. Therefore, y is the average of x and z. □

This problem suggests why the average of a group of numbers is also sometimes called the **arithmetic mean** of the numbers.

Try to solve the following problem first by using the formula we developed earlier, then again using your understanding of arithmetic sequences (in other words, without using the formula).

Problem 21.3: The sequence

$$4, x_1, x_2, x_3, x_4, 18$$

is an arithmetic sequence. Find x_3.

Solution for Problem 21.3: We offer a "formula" solution and an "intuitive" solution.

Solution 1: Use the formula. We are given the first term, so we already know $a = 4$. We are also given the sixth term. Letting the common difference between terms be d, our sixth term gives us the equation

$$4 + (6-1)d = 18.$$

Solving this equation gives $d = \frac{14}{5}$. We must find x_3, which is the fourth term in the sequence. Our formula gives

$$x_3 = 4 + (4-1)d = 4 + 3\left(\frac{14}{5}\right) = \frac{62}{5}.$$

Solution 2: Use our understanding of arithmetic sequences. The 18 at the end of the sequence is 5 steps from 4. These 5 steps cover a distance of $18 - 4 = 14$, so each step has length $\frac{14}{5}$. We must take three such steps to get from the first term to x_3, so

$$x_3 = 4 + 3\left(\frac{14}{5}\right) = \frac{62}{5}.$$

Notice that our solutions are essentially the same. □

WARNING!! Don't simply memorize the formulas in this chapter. If you take the time to understand them, you'll be able to solve problems much more quickly with much less likelihood of making a mistake. Also, once you understand the formulas, you won't have to memorize them – you'll simply know them.

And once you do know them, you'll have no difficulty tackling problems like the following one.

Problem 21.4: The sum of the second term and the ninth term of an arithmetic sequence is -4. The sum of the third and fourth terms of the same sequence is 4. Find the first term of the sequence.

Solution for Problem 21.4: Letting the first term be a and the common difference be d, we have

$$\begin{aligned} \text{Second term} &= a + d, \\ \text{Ninth term} &= a + 8d, \\ \text{Third term} &= a + 2d, \\ \text{Fourth term} &= a + 3d. \end{aligned}$$

Using these expressions with the given information about sums of these terms, we have the equations

$$\begin{aligned} (a + d) + (a + 8d) &= -4, \\ (a + 2d) + (a + 3d) &= 4. \end{aligned}$$

Simplifying the left hand sides gives the equations

$$\begin{aligned} 2a + 9d &= -4, \\ 2a + 5d &= 4. \end{aligned}$$

Subtracting the second equation from the first gives $4d = -8$, so $d = -2$. Substituting this into either of our equations gives $a = 7$, so the first term of the sequence is 7. □

Exercises

21.1.1 Consider the arithmetic sequence $1, 4, 7, 10, 13, \ldots$

(a) Find the 15^{th} term in the sequence.

(b) Find a formula for the n^{th} term in the sequence.

21.1.2 The third term of an arithmetic sequence is 5 and the sixth term is -1. Find the twelfth term of this sequence.

21.1.3 How many terms are in the arithmetic sequence $5, 11, 17, \ldots, 89$?

21.1.4 When the 171^{st} even positive integer is subtracted from the 219^{th} odd positive integer, the result is z. Find z. *(Source: MATHCOUNTS)*

21.1.5★ In the infinite arithmetic sequence $a_1, a_2, a_3, \ldots$, we have $a_8 = 2001$. If the common difference d is an integer, find the minimum value of d so that $a_{17} > 10000$. **Hints:** 5

21.2 Arithmetic Series

When we add a group of consecutive terms of an arithmetic sequence, we form an **arithmetic series.** For example, the series

$$1 + 2 + 3 + 4 + \cdots + 99 + 100$$

is an arithmetic series.

Problems

Problem 21.5: According to legend, the great mathematician Carl Gauss was given the busy-work assignment in elementary school of finding the sum of the first 100 positive integers. (Busy work is not a recent invention – Gauss was born in 1777.) While the other students scribbled away tediously adding and adding and adding, Gauss thought briefly, then wrote down the correct answer. In this problem, we re-create the method he allegedly used to find the sum:

$$1 + 2 + 3 + 4 + \cdots + 99 + 100.$$

(a) Write the sum backwards, starting with 100 and ending at 1.

(b) Add the series you wrote in part (a) to the original series by summing the first terms of each series, then summing the second terms of each series, then summing the third terms, and so on. Do you see anything interesting in your sums?

(c) Find the sum $1 + 2 + 3 + \cdots + 99 + 100$.