Don't worry about the fancy name – as we saw in Problem 20.19, partial fraction decomposition isn't very hard to do. In Section 21.5★, we'll see an interesting use of partial fraction decomposition. When you get to the study of calculus, you'll see several more applications.

**Exercises**

**20.4.1** Solve the equation $\frac{2x}{x-5} = 3 + \frac{1-x}{x-3}$.

**20.4.2** Consider the function $g(x) = \frac{3-2x}{x-7}$.

(a) Find the domain of $g$.

(b) Find the range of $g$.

(c) Find all horizontal and vertical asymptotes of the graph of $y = g(x)$.

(d) Graph $y = g(x)$.

**20.4.3** Find constants $A$ and $B$ such that $\frac{x+7}{x^2-2x-35} = \frac{A}{x-7} + \frac{B}{x+5}$ for all $x$.

**20.4.4** Consider the function

$$g(x) = \frac{3x^2 - 7x + 4}{x^2 + 4x - 5}.$$

(a) Factor the numerator and denominator.

(b) What is the domain of $g$?

(c) How does $g$ differ from the function $f(x) = \frac{3x-4}{x+5}$ in Problem 20.18?

(d) What point must be omitted from the graph of $f$ in Problem 20.18 to produce the graph of $g$?

(e) What is the range of $g$?

## 20.5 Piecewise Defined Functions

We've described functions as machines that take an input, then produce an output according to some rule. We could also imagine a machine that takes an input, but applies different rules to different inputs in order to produce an output. We call such a function a **piecewise defined function**, and to describe such a function we must give both the rules we use to produce output, and describe which inputs each rule is for. Here is an example of how we usually do so:

$$f(x) = \begin{cases} 2x & \text{if } x < 0, \\ 3x & \text{if } x > 0. \end{cases}$$

This describes a function that doubles negative inputs and triples positive inputs. Notice that $f(x)$ is not defined for $x = 0$, so 0 is not in the domain of $f$.

**Problems**

**Problem 20.20:** Let

$$f(x) = \begin{cases} x+5 & \text{if } x \le -3, \\ 2x^2 - 1 & \text{if } -3 < x \le 2, \\ -x/2 & \text{if } x > 2. \end{cases}$$

(a) Find $f(1)$, $f(9)$, and $f(-12)$.
(b) Find $f(f(f(f(-12))))$.
(c) For what values of $x$ is $f(x) = x$?
(d) Does $f$ have an inverse?
(e) Graph $y = f(x)$.

**Problem 20.21:**
(a) Write the function $f(x) = |x|$ as a piecewise defined function without using absolute value signs.
(b) Write the function $f(x) = |5 - x|$ as a piecewise defined function without using absolute value signs.
(c) Write the function $f(x) = |-3 - 7x|$ as a piecewise defined function without using absolute value signs.

**Problem 20.22:** A function $f$ from the integers to the integers is defined as follows:

$$f(n) = \begin{cases} n+3 & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

Suppose $k$ is odd and $f(f(f(k))) = 27$. Find $k$. *(Source: AHSME)*

We start as we have before with new functions: experimenting with and graphing a sample function.

**Problem 20.20:** Let

$$f(x) = \begin{cases} x+5 & \text{if } x \le -3, \\ 2x^2 - 1 & \text{if } -3 < x \le 2, \\ -x/2 & \text{if } x > 2. \end{cases}$$

(a) Find $f(1)$, $f(9)$, and $f(-12)$.
(b) Find $f(f(f(f(-12))))$.
(c) For what values of $x$ is $f(x) = x$?
(d) Does $f$ have an inverse?
(e) Graph $y = f(x)$.

*Solution for Problem 20.20:*

(a) Since 1 is between $-3$ and 2, we use our middle rule to compute $f(1) = 2(1^2) - 1 = 1$. Since $9 > 2$, we use our bottom rule to find $f(9) = -9/2$, and since $-12 \le -3$, we use our first rule to compute $f(-12) = -12 + 5 = -7$.

(b) We already found $f(-12) = -7$, so we have $f(f(f(f(-12)))) = f(f(f(-7)))$. Since $-7 \le -3$, we have $f(-7) = -7 + 5 = -2$, so $f(f(f(-7))) = f(f(-2))$. Since $-2$ is between $-3$ and 2, we have $f(-2) = 2(-2)^2 - 1 = 7$, and $f(f(-2)) = f(7)$. Finally, $7 > 2$, so $f(7) = -7/2$. Putting this all together gives

$$f(f(f(f(-12)))) = f(f(f(-7))) = f(f(-2)) = f(7) = -7/2.$$

(c) We wish to find the solutions to the equation $f(x) = x$. First, we check for solutions where $x \le -3$. These must satisfy the equation $x + 5 = x$ because $f(x) = x + 5$ if $x \le -3$. There are no solutions to the equation $x + 5 = x$, so there are no values of $x$ such that $x \le -3$ and $f(x) = x$.

Next, we check for solutions between $-3$ and 2. These must satisfy $2x^2 - 1 = x$ because $f(x) = 2x^2 - 1$ when $-3 < x \le 2$. Rearranging $2x^2 - 1 = x$ gives $2x^2 - x - 1 = 0$ and factoring gives $(2x + 1)(x - 1) = 0$. The solutions to this equation are $x = 1$ and $x = -1/2$, which are both between $-3$ and 2.

Finally, we look for solutions that are greater than 2. These must satisfy $-x/2 = x$. The only solution to this equation is $x = 0$, but 0 is not greater than 2. We only have $f(x) = -x/2$ if $x > 2$, and the only values of $x$ for which $-x/2 = x$ are less than 2, so there are no values of $x$ such that $x > 2$ and $f(x) = x$.

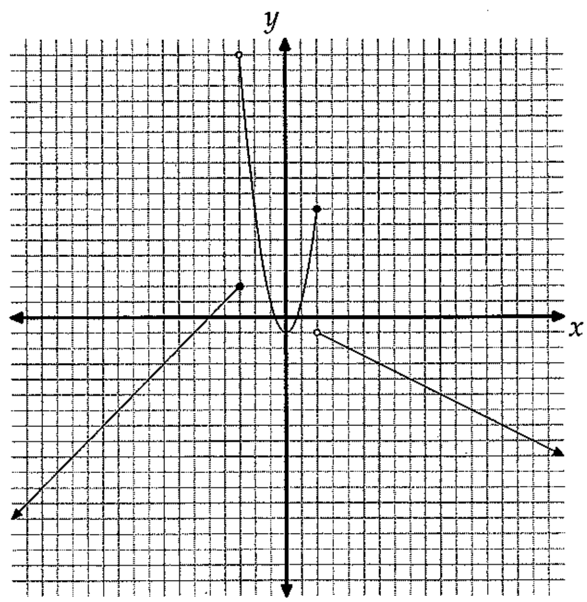
Therefore, the only values of $x$ for which $f(x) = x$ are $x = 1$ and $x = -1/2$.

(d) If $f$ gives the same output for two different inputs, then we know that $f$ does not have an inverse. One example of such an output is 1. Because

$$f(-1) = 2(-1)^2 - 1 = 1 \qquad \text{and} \qquad f(1) = 2(1)^2 - 1 = 1,$$

there is more than one value of $x$ such that $f(x) = 1$, so $f$ does not have an inverse. As we'll see below, we can also use the graph of $f$ to quickly see that $f$ does not have an inverse.

(e) We graph $y = f(x)$ by graphing each of its pieces, using the values of $x$ for which $f(x)$ is defined for each piece.



In our graph, the open circles mark points that our graph approaches but never quite reaches, and the solid circles mark points that are on the graph. We can clearly see the three pieces of our function. We can also see that this function cannot have an inverse because there are many horizontal lines that pass through more than one point on the graph.

The gaps in our graph indicate that the function is not **continuous**, which basically means we can't draw the graph of the function without lifting our pencil off the paper.

□

Although we haven't written absolute value as a piecewise defined function, we have treated it that way when we broke absolute value problems into cases. We can use the piecewise defined function notation to indicate these cases.

**Problem 20.21:** Write the function $f(x) = |-3 - 7x|$ as a piecewise defined function without using absolute value signs.

*Solution for Problem 20.21:* Rather than start with such a complicated expression, we get a handle on the problem by trying simpler expressions first.

**Concept:** Trying simpler versions of a complicated problem can often guide you to a solution to the problem.

Let's try expressing the simplest absolute value function, $g(x) = |x|$, as a piecewise defined function. Our two cases are $x \geq 0$ and $x < 0$. When $x \geq 0$, we have $g(x) = |x| = x$ and when $x < 0$, we have $g(x) = |x| = -x$. Therefore, we can write

$$g(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Let's try a slightly harder function,

$$h(x) = |5 - x|.$$

As before, we use casework. One case consists of the values of $x$ that make the expression inside the absolute value nonnegative and the other consists of those values of $x$ that make this expression negative.

- *Case 1:* $5 - x \geq 0$. When $5 - x \geq 0$, we have $x \leq 5$ and $|5 - x| = 5 - x$.
- *Case 2:* $5 - x < 0$. When $5 - x < 0$, we have $x > 5$ and $|5 - x| = -(5 - x) = x - 5$.

We can read the pieces of our piecewise defined function directly from these cases:

$$h(x) = \begin{cases} 5 - x & \text{if } x \leq 5, \\ -5 + x & \text{if } x > 5. \end{cases}$$

Now, we're ready to tackle the original problem. We split the function into cases based on when $-3 - 7x$ is negative and when it is nonnegative.

- *Case 1:* $-3-7x \geq 0$. If $-3-7x \geq 0$, then $x \leq -\frac{3}{7}$, and $|-3-7x| = -3-7x$.
- *Case 2:* $-3-7x < 0$. If $-3-7x < 0$, then $x > -\frac{3}{7}$, and $|-3-7x| = -(-3-7x) = 3+7x$.

Converting our cases to piecewise defined notation, we have

$$f(x) = \begin{cases} -3-7x & \text{if } x \leq -\dfrac{3}{7}, \\ 3+7x & \text{if } x > -\dfrac{3}{7}. \end{cases}$$

□

**Problem 20.22:** A function $f$ from the integers to the integers is defined as follows:

$$f(n) = \begin{cases} n+3 & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

Suppose $k$ is odd and $f(f(f(k))) = 27$. Find $k$. *(Source: AHSME)*

*Solution for Problem 20.22:* In order to find $k$, we have to "undo" the function three times. First, we need to find what inputs can produce 27 as output from $f$. This means we must find the values of $n$ such that $f(n) = 27$. We must check two possibilities:

- *Case 1: $n$ is odd.* If $n$ is odd, then $f(n) = n + 3$. So, if $n$ is odd and $f(n) = 27$, we must have $n + 3 = 27$. Solving this gives $n = 24$. However, 24 is not odd, so there are no odd numbers $n$ such that $f(n) = 27$.
- *Case 2: $n$ is even.* If $n$ is even, then $f(n) = n/2$. So, if $n$ is even and $f(n) = 27$, then $n/2 = 27$. Solving this gives $n = 54$. Since 54 is even, we have found a number $n$ such that $f(n) = 27$.

Therefore, we have $f(54) = 54/2 = 27$, so we can "undo" one step. We now seek the odd value of $k$ such that

$$f(f(k)) = 54,$$

since we will then have the value of $k$ for which $f(f(f(k))) = f(54) = 27$.

Now we must determine how we can produce 54 from $f(n)$. Again, we consider the cases "$n$ is odd" and "$n$ is even" separately. This time, both cases provide solutions. The "$n$ is odd" case gives us $f(51) = 54$ and the "$n$ is even" case gives $f(108) = 54$. So, we now seek values of $k$ such that $f(k)$ equals either 51 or 108, since in either case we have $f(f(k)) = 54$.

For $f(k) = 51$, we find that we have no solutions for the "$n$ is odd" piece, because 51 cannot equal the sum of 3 and an odd number. The even case gives us $f(102) = 51$, but we are told that $k$ is odd, so we discard this solution..

Turning to $f(k) = 108$, we check out the odd case first. Since $f(105) = 105 + 3 = 108$, and $k = 105$ is odd, we have our desired odd $k$. (Note we don't have to check the even case, since we are told that $k$ is odd.) Checking our answer, we find that $f(f(f(105))) = f(f(108)) = f(54) = 27$. □

## Exercises

**20.5.1** Let $f(x) = \begin{cases} 2x+9 & \text{if } x < -2, \\ 5-2x & \text{if } x \geq -2. \end{cases}$

(a) Find $f(3)$.

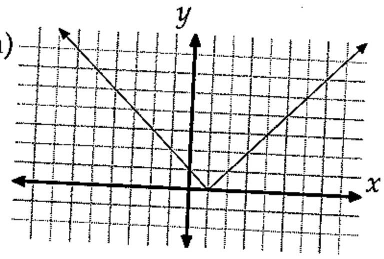
(b) Find $f(-7)$.

(c) Find all values of $k$ such that $f(k) = 3$.

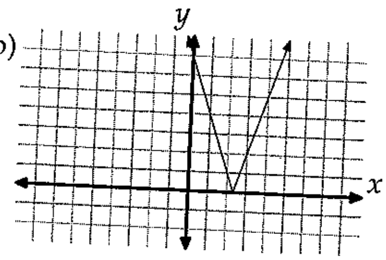
(d) Does $f(x)$ have an inverse?

**20.5.2** Each graph below is the graph of a function. For each part, define the function as a piecewise defined function, then again using absolute value.

(a)




(b)




**20.5.3** Mientka Publishing Company prices its bestseller *Where's Walter?* as follows:

$$C(n) = \begin{cases} 12n, & \text{if } 1 \leq n \leq 24, \\ 11n, & \text{if } 25 \leq n \leq 48, \\ 10n, & \text{if } 49 \leq n, \end{cases}$$

where $n$ is the number of books ordered, and $C(n)$ is the cost in dollars of $n$ books. Notice that 25 books cost less than 24 books. For how many values of $n$ is it cheaper to buy more than $n$ books than to buy exactly $n$ books? *(Source: AHSME)*

**20.5.4** Let

$$f(x) = \begin{cases} 2x^2 - 3 & \text{if } x \leq 2, \\ ax+4 & \text{if } x > 2. \end{cases}$$

Find $a$ if the graph of $y = f(x)$ is continuous (which means the graph can be drawn without lifting your pencil from the paper).

**20.5.5★** In Problem 20.21, we wrote a function expressed in terms of absolute value as a piecewise defined function. In this problem, we try to reverse this process, taking a function that is expressed as a piecewise defined function and expressing it using absolute value instead. Write the function

$$f(x) = \begin{cases} -x-1 & \text{if } x < -3, \\ x+5 & \text{if } x \geq -3. \end{cases}$$

using absolute value rather than using piecewise defined function notation.