20.2.5 Solve the equation $|2z - 9| + |z - 3| = 15$.

20.2.6 Graph each of the following on the Cartesian plane.

(a) $y = |x + 4|$

(b) $y = |3 - x|$

(c) $y = |x + 4| + |3 - x|$

(d) $|y - 2| < 4$

20.2.7★ Solve the equation $|r^2 - 5r| = 6$.

20.3 Floor and Ceiling

We have special notation that means "round down": $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x. We call $f(x) = \lfloor x \rfloor$ the **floor function**. It is also sometimes called the **greatest integer function**, and sometimes denoted $[x]$.

Some examples of the floor function in action are

$$\lfloor 2.3 \rfloor = 2, \qquad \lfloor 7 \rfloor = 7, \qquad \lfloor -5.3 \rfloor = -6.$$

That last example merits special attention.

WARNING!! Be careful when applying the floor function to negative numbers. Rounding down doesn't just mean removing the numbers after the decimal point.

In words, we often say $\lfloor x \rfloor$ is "the floor of x." For example, the floor of 2.3 is 2.

To "round up," we use the **ceiling function**, $f(x) = \lceil x \rceil$. This function returns the smallest integer that is greater than or equal to x. So, for example,

$$\lceil 2.3 \rceil = 3, \qquad \lceil 7 \rceil = 7, \qquad \lceil -5.3 \rceil = -5.$$

Once again, we have to be careful about negative numbers!

Problems

Problem 20.12: Evaluate each of the following:

(a) $\lfloor 2.7 \rfloor$

(b) $\lceil 3.5 \rceil$

(c) $\lceil -2.3 \rceil$

(d) $\lfloor \frac{123}{5} \rfloor$

(e) $\lceil \sqrt{39} \rceil$

(f) $\left\lfloor \sqrt{\lfloor \sqrt{345} \rfloor} \right\rfloor$ *(Source: Mandelbrot)*

Problem 20.13: Let $f(x) = \lfloor x \rfloor$ and $g(x) = \lceil x \rceil$. Graph the equations $y = f(x)$ and $y = g(x)$.

Problem 20.14: For what values of x is $\lceil x \rceil = \lfloor x \rfloor$?

Problem 20.15: The notation $\{x\}$ is sometimes used to denote the **fractional part** of x, which means the smallest amount that can be subtracted from x to produce an integer. For example, $\{4.3\} = 0.3$ and $\{-5.6\} = 0.4$, since -5.6 is 0.4 more than an integer. Write an equation relating $\lfloor x \rfloor$, x, and $\{x\}$.

Problem 20.16: In this problem, we compute the number of ordered pairs (x, y) with $x > 0$ and $y > 0$ such that (x, y) satisfies the system of equations

$$x + \lfloor y \rfloor = 5.3,$$
$$y + \lfloor x \rfloor = 5.7.$$

(Source: ARML)

(a) What does the first equation tell us about the fractional part of x?

(b) What does the second equation tell us about the fractional part of y?

(c) Solve the problem.

We start by evaluating the floor and ceiling of a few numbers.

Problem 20.12: Evaluate each of the following:

(a) $\lfloor 2.7 \rfloor$

(b) $\lceil 3.5 \rceil$

(c) $\lceil -2.3 \rceil$

(d) $\lfloor \frac{123}{5} \rfloor$

(e) $\left\lceil \sqrt{39} \right\rceil$

(f) $\left\lfloor \sqrt{\left\lfloor \sqrt{345} \right\rfloor} \right\rfloor$ *(Source: Mandelbrot)*

(a) 2.7 rounds down to 2, so $\lfloor 2.7 \rfloor = 2$.

(b) 3.5 rounds up to 4, so $\lceil 3.5 \rceil = 4$.

(c) The smallest integer that is greater than -2.3 is -2, so -2.3 rounds up to -2.

(d) We write $123/5$ as a decimal to make rounding it easier: $123/5 = 24.6$, so $\lfloor \frac{123}{5} \rfloor = \lfloor 24.6 \rfloor = 24$.

(e) We need to figure out what two integers $\sqrt{39}$ is between to find out how to round it up. Because $6^2 = 36 < 39 < 49 = 7^2$, we know that $\sqrt{39}$ is between 6 and 7. So, $\left\lceil \sqrt{39} \right\rceil = 7$.

(f) We work from the inside out. First, because $18^2 = 324 < 345 < 361 = 19^2$, we have $18 < \sqrt{345} < 19$. Therefore, we know that $\left\lfloor \sqrt{345} \right\rfloor = 18$, which means

$$\left\lfloor \sqrt{\left\lfloor \sqrt{345} \right\rfloor} \right\rfloor = \left\lfloor \sqrt{18} \right\rfloor.$$

Since 18 is between 4^2 and 5^2, we have $4 < \sqrt{18} < 5$, so $\left\lfloor \sqrt{18} \right\rfloor = 4$.

Important: When it isn't immediately obvious what the floor or ceiling of a number is, then try to find the two consecutive integers that the number is between.

To further our understanding of the floor and ceiling functions, let's graph them.

Problem 20.13: Let $f(x) = \lfloor x \rfloor$ and $g(x) = \lceil x \rceil$. Graph the equations $y = f(x)$ and $y = g(x)$.

Solution for Problem 20.13: We'll start with $y = f(x) = \lfloor x \rfloor$. We build our graph by considering various values of x. For example, for $0 \le x < 1$, we have $f(x) = 0$. Then, for $1 \le x < 2$, we step up to $f(x) = 1$. For $2 \le x < 3$, we have $f(x) = 2$. Continuing in this manner, we find that the graph of $y = \lfloor x \rfloor$ is a series of steps as shown at left below. The open circles indicate lattice points that are not on the graph and the closed circles indicate lattice points that are on the graph. The steps continue forever in both directions.

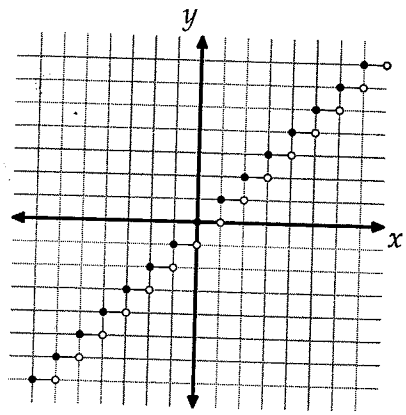


Figure 20.4: Graph of $y = \lfloor x \rfloor$

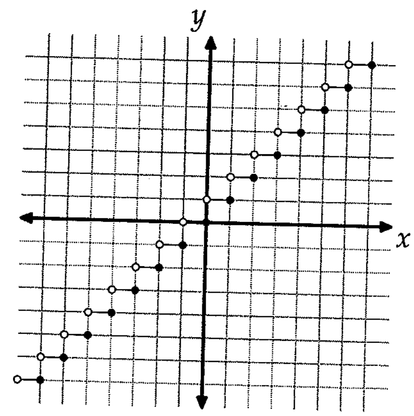


Figure 20.5: Graph of $y = \lceil x \rceil$

We similarly explore $y = g(x) = \lceil x \rceil$. For $0 < x \le 1$, we have $g(x) = 1$. Then for $1 < x \le 2$, we step up to $g(x) = 2$. Continuing this way, we build the graph at the right above. Notice that this is almost a 1-unit upward shift of the graph of $y = \lfloor x \rfloor$ (but not exactly a 1-unit upward shift: look at the lattice points). □

Problem 20.14: For what values of x is $\lceil x \rceil = \lfloor x \rfloor$?

Solution for Problem 20.14: Our graphs from the previous problem show that the only values of x for which $\lceil x \rceil = \lfloor x \rfloor$ are the integers. We can also see this by noting that if x is an integer, then $\lceil x \rceil = \lfloor x \rfloor = x$, but if x is not an integer, then $\lceil x \rceil > x > \lfloor x \rfloor$ because we round up to get the ceiling and down to find the floor. □

The notation $\{x\}$ is sometimes used to denote the fractional part of x, which is the smallest amount that can be subtracted from x to produce an integer. For example, we have $\{4.3\} = 0.3$, because 4.3 is 0.3 more than 4, and $\{-5.6\} = 0.4$, since -5.6 is 0.4 more than -6. Notice that we must have $0 \le \{x\} < 1$ for all real numbers x, because there is always some nonnegative number less than 1 that we can subtract from x to get an integer.

Problem 20.15: Write an equation relating $\lfloor x \rfloor$, x, and $\{x\}$.

Solution for Problem 20.15: To get a feel for the problem, let's try a couple specific values of x. Suppose $x = 3.14$. Then, $\lfloor x \rfloor = \lfloor 3.14 \rfloor = 3$ and $\{x\} = \{3.14\} = 0.14$. So, we can view $\lfloor 3.14 \rfloor$ and $\{3.14\}$ as breaking 3.14 into two parts: what we get when we round x down, and what we remove in order to round x

down. If we put these parts back together, we get 3.14:

$$\lfloor 3.14 \rfloor + \{3.14\} = 3.14.$$

Let's see if this works for a negative number, as well. Suppose $x = -2.7$. Then, we have $\lfloor -2.7 \rfloor = -3$ and $\{-2.7\} = 0.3$. As we did before, we can assemble these parts to get x back:

$$\lfloor -2.7 \rfloor + \{-2.7\} = -2.7.$$

This relationship holds for all x. Because we "round down" to find the floor of x, the floor of x is what's left after the fractional part of x is removed. In other words, to find the floor of x, we subtract the fractional part of x from x:

$$\lfloor x \rfloor = x - \{x\}.$$

Adding $\{x\}$ to both sides of this equation gives us the relationship we saw in our two examples:

$$\lfloor x \rfloor + \{x\} = x.$$

□

Concept: Some problems involving $\lfloor x \rfloor$ can be tackled by thinking of x as the sum of its floor and its fractional part:

$$x = \lfloor x \rfloor + \{x\}.$$

Try this tactic on the following problem.

Problem 20.16: Compute the number of ordered pairs (x, y) with $x > 0$ and $y > 0$ such that we have

$$x + \lfloor y \rfloor = 5.3,$$
$$y + \lfloor x \rfloor = 5.7.$$

(Source: ARML)

Solution for Problem 20.16: We can't solve this as an ordinary system of linear equations because of the floor functions. Instead, we'll have to think more carefully about what each equation means. We start with the first:

$$x + \lfloor y \rfloor = 5.3.$$

This equation tells us that x plus some integer equals 5.3. While this doesn't tell us x, it does tell us that the fractional part of x is 0.3. We can also see this by letting $x = \lfloor x \rfloor + \{x\}$:

$$\lfloor x \rfloor + \{x\} + \lfloor y \rfloor = 5.3.$$

Since $\lfloor x \rfloor$ and $\lfloor y \rfloor$ are integers, the .3 of 5.3 must come from the fractional part of x. Therefore, we have $\{x\} = 0.3$. (Remember, the fractional part is always greater than or equal to 0 and less than 1. So, for example, we can't have $\{x\} = 1.3$.) Letting $\{x\} = 0.3$ in our equation above gives us

$$\lfloor x \rfloor + \lfloor y \rfloor = 5.$$

Using the same analysis on the equation

$$y + \lfloor x \rfloor = 5.7$$

tells us that the fractional part of y is 0.7, and that, once again, we have

$$\lfloor x \rfloor + \lfloor y \rfloor = 5.$$

Because x and y are positive and their floors are integers, our only solutions to this equation are

$$(\lfloor x \rfloor, \lfloor y \rfloor) = (0,5);\ (1,4);\ (2,3);\ (3,2);\ (4,1);\ (5,0).$$

Including the fractional parts of x and y gives the six solutions

$$(x, y) = (0.3, 5.7);\ (1.3, 4.7);\ (2.3, 3.7);\ (3.3, 2.7);\ (4.3, 1.7);\ (5.3, 0.7).$$

□

Exercises

20.3.1 Evaluate each of the expressions below.

(a) $\lfloor 3.2 \rfloor$

(b) $\lfloor -21.8 \rfloor$

(c) $\left\lceil \frac{230}{7} \right\rceil$

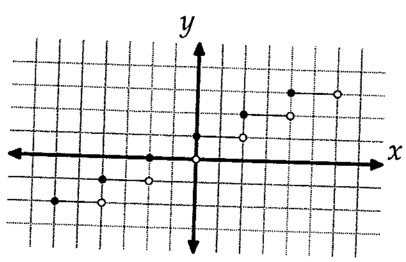
(d) $\left\lceil -\sqrt{23} \right\rceil$

(e)★ $\left\lfloor \sqrt{26} - \sqrt{8} \right\rfloor$

(f)★ $\left\lfloor \sqrt{\frac{104}{5}} \right\rfloor$

20.3.2 Compute $\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \cdots + \lfloor \sqrt{16} \rfloor$. *(Source: AMC 12)*

20.3.3★ Use the floor function to write a function $f(x)$ such that the graph of $y = f(x)$ for $-6 \le x < 6$ is shown at right. **Hints:** 3



20.3.4★ Find the sum of the three smallest positive solutions to the equation $x - \lfloor x \rfloor = \frac{1}{\lfloor x \rfloor}$. *(Source: MATHCOUNTS)* **Hints:** 145

20.4 Rational Functions

We call a function that is the ratio of two polynomials a **rational function**. Here are some examples of rational functions:

$$f(x) = \frac{x+4}{x^2 - 4x + 3}, \qquad g(y) = \frac{2 - y^2}{y}, \qquad h(z) = \frac{3z^4 + 2z^3 - 1}{6z + 1}.$$