Problem 20.5: Find all solutions to the equation $\sqrt{2x+10} - \sqrt{7-x} = \sqrt{2x-2}$.

Solution for Problem 20.5: We start by squaring to get rid of some of the radicals. On the left side we have

$$\begin{aligned}\left(\sqrt{2x+10} - \sqrt{7-x}\right)^2 &= \left(\sqrt{2x+10}\right)^2 - 2\sqrt{2x+10}\sqrt{7-x} + \left(\sqrt{7-x}\right)^2\\ &= 2x+10-2\sqrt{(2x+10)(7-x)}+7-x\\ &= 17+x-2\sqrt{-2x^2+4x+70}.\end{aligned}$$

On the right we simply have $\left(\sqrt{2x-2}\right)^2 = 2x-2$, so our equation now is

$$17+x-2\sqrt{-2x^2+4x+70} = 2x-2.$$

Now we can isolate the radical, which gives

$$2\sqrt{-2x^2+4x+70} = 19-x.$$

Squaring this equation gives us $4(-2x^2+4x+70) = (19-x)^2$. Expanding both sides then gives us the equation $-8x^2+16x+280 = 361-38x+x^2$, and rearranging this results in

$$9x^2-54x+81=0.$$

Dividing this equation by 9 gives us $x^2-6x+9=0$. Factoring gives $(x-3)^2=0$, so our only possible solution is $x=3$. Substituting $x=3$ into the original equation gives us $4-2=2$. Therefore, the only solution to the original equation is $x=3$. □

Exercises

20.1.1 What are the domain and range of each of the following functions?

(a) $f(x) = \sqrt{2+x} - 5$ (b) $g(x) = -2\sqrt{3-x}+7$

20.1.2 Find all solutions to the equation $\sqrt{1+8r-r^2} = 4$.

20.1.3 If $x > 0$, then simplify $\sqrt{\dfrac{x}{1-\frac{x-1}{x}}}$. *(Source: AMC 12)*

20.1.4 Graph the equation $y = -\sqrt{x-2}$.

20.1.5 If $\sqrt{2+\sqrt{x}} = 3$, then what is x? *(Source: AHSME)*

20.1.6★ Solve the equation $\sqrt{x}+\sqrt{x+4} = 2\sqrt{4x-5}$.

20.2 Absolute Value

The **absolute value** of a number can be thought of as its distance from 0 on the number line. For example, $|5| = 5$, because 5 is 5 units from 0 on the number line. The number -5 is also 5 units away from 0 on the number line, so $|-5| = 5$, as well.

We can also use absolute value to express the distance between two different numbers on the number line. Specifically, the expression $|x - y|$ equals the distance between x and y on the number line. For example, we have $|6 - (-3)| = 9$ because 6 and -3 are 9 apart on the number line.

Problems

Problem 20.6:
(a) If $x \geq 0$, then must $|x|$ equal x? Why or why not?
(b) If $x < 0$, then must $|x|$ equal $-x$? Why or why not?
(c) Graph the equation $y = |x|$.

Problem 20.7: In this problem, we graph the equation $y = |2x + 5| - 3$.
(a) For what values of x is $2x + 5$ nonnegative? What linear expression does $|2x + 5| - 3$ equal for these values of x?
(b) For what values of x is $2x + 5$ negative? What linear expression does $|2x + 5| - 3$ equal for these values of x?
(c) Use your observations in the first two parts to draw the graph of $y = |2x + 5| - 3$.

Problem 20.8: In this problem, we find all values of x such that

$$|2x - 9| = 5$$

in several different ways.
(a) *Method 1:* If $|2x - 9| = 5$, then what are the two possible values of $2x - 9$?
(b) *Method 2:* For what values of x is $2x - 9 \geq 0$? How can we rewrite the equation when x is in this range? What solution does this give?
(c) For what values of x is $2x - 9 < 0$? How can we rewrite the equation when x is in this range? What solution does this give?
(d) *Method 3:* Why is $|x|^2 = x^2$? Does this suggest another solution to this problem?
(e) *Method 4:* If $|2x - 9| = 5$, then how far apart are $2x$ and 9 on the number line? Does this suggest yet another solution to this problem?

Problem 20.9: Consider the function

$$f(x) = |x - 3| + |x + 2|.$$

(a) What linear expression does $f(x)$ equal if $x \geq 3$?
(b) If $-2 \leq x < 3$, then how can we write $|x - 3|$ without absolute value signs? How can we write $|x + 2|$ without absolute value signs? How can we write $f(x)$ when x is in this range?
(c) What linear expression does $f(x)$ equal if $x < -2$?
(d) Graph $y = f(x)$.

Problem 20.10: Find all solutions to the equation $|x+3|+|2-5x|=7$.

Problem 20.11: Graph the equation $|x+1|+|y-2|=5$.

Problem 20.6:
(a) If $x \geq 0$, then must $|x|$ equal x? Why or why not?
(b) If $x < 0$, then must $|x|$ equal $-x$? Why or why not?
(c) Graph the equation $y = |x|$.

Solution for Problem 20.6:

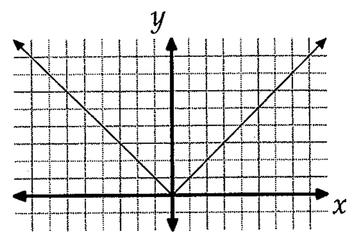
(a) When x is nonnegative, the absolute value of x is just x, since x is x units from 0 on the number line.

(b) When x is negative, we can't say that $|x| = x$, because distance must be positive. If x is negative, then it is a distance of $-x$ from 0. So, if x is negative, then $|x| = -x$.

Important: If $x \geq 0$, then $|x| = x$. If $x < 0$, then $|x| = -x$.

(c) When x is positive, $f(x) = |x|$ is exactly the same as $f(x) = x$. When it is negative, $f(x) = |x|$ is exactly the same as $f(x) = -x$.

We're ready to graph $y = |x|$. For nonnegative x (to the right of the y-axis), we graph $y = x$, and for negative x (to the left of the y-axis) we graph $y = -x$. The resulting graph is shown at right.



□

Our graph of $y = |x|$ reinforces a very important property of absolute value:

Important: The absolute value of a real number is always nonnegative.

Now that we can graph $y = |x|$, let's try graphing a more complicated equation involving absolute value.

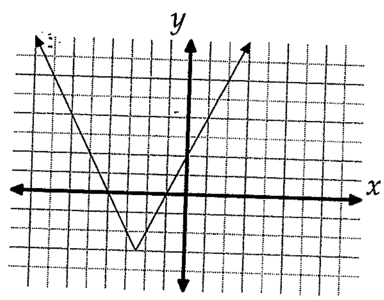
Problem 20.7: Graph the equation $y = |2x+5| - 3$.

Solution for Problem 20.7: We graphed $y = |x|$ by considering the cases $x \geq 0$ and $x < 0$. These cases allowed us to get rid of the absolute value symbol, since $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x < 0$. We try the same thing here.

First, we have to determine what our cases are. The absolute value of a nonnegative number is the number itself, so one case is when the expression inside the absolute value symbol in the equation $y = |2x+5| - 3$ is nonnegative. So, for this case we have $2x + 5 \geq 0$. On the other hand, the absolute value of a negative number is the opposite of the number. Therefore, our other case is when $2x + 5$ is negative. Now, we're ready to consider our two cases:

- *Case 1: $2x + 5$ is nonnegative.* When $2x + 5 \geq 0$, we have $x \geq -5/2$. We also have $|2x + 5| = 2x + 5$ when $2x + 5 \geq 0$, so our equation becomes $y = 2x + 5 - 3 = 2x + 2$. So, when $x \geq -5/2$, the equation $y = |2x + 5| - 3$ is the same as the equation $y = 2x + 2$.
- *Case 2: $2x+5$ is negative.* When $2x+5 < 0$, we have $x < -5/2$. We also have $|2x+5| = -(2x+5) = -2x-5$ when $2x + 5 < 0$, so our equation in this case becomes $y = |2x + 5| - 3 = -2x - 5 - 3 = -2x - 8$. Therefore, when $x < -5/2$ the equation $y = |2x + 5| - 3$ is the same as the equation $y = -2x - 8$.

Putting these two cases together, we have the graph shown at right. For $x \geq -5/2$, our graph is the same as the graph of $y = 2x + 2$, as determined above. For $x < -5/2$, our graph is the same as the graph of $y = -2x - 8$.



The graph of a function that consists of the absolute value of a linear expression, such as $f(x) = |x|$ or $g(x) = |2x - 5|$, always has a V-shape like the one shown at right. Notice also that the slopes of the two branches of the V are 2 and -2. Our casework above shows why the absolute values of the slopes of our two branches must equal the coefficient of x in $y = |2x + 5| - 3$. □

Important: Whenever you see a V-shaped graph (or an upside-down V), you should think of absolute value.

Problem 20.8: Find all values of x such that $|2x - 9| = 5$.

Solution for Problem 20.8: We present several solutions.

Solution 1: What can $|2x - 9|$ equal? If $|2x - 9| = 5$, then $2x - 9$ must be either 5 or -5, because 5 and -5 are the only numbers that have absolute value equal to 5. If $2x - 9 = 5$, then $x = 7$. If $2x - 9 = -5$, then $x = 2$. So, our solutions are $x = 2$ and $x = 7$.

While this approach is easy for a simple equations like $|2x - 9| = 5$, we will soon see more complex absolute value equations for which this approach won't work. For these more complicated problems, we can often use one of the following methods:

Solution 2: Casework. We don't know how to solve equations with absolute value signs, so we want to get rid of them. We know that when $2x - 9$ is nonnegative, then $|2x - 9| = 2x - 9$. When $2x - 9$ is negative, then $|2x - 9| = -(2x - 9)$. Make sure you see why! We must therefore consider two cases:

- *Case 1:* $2x - 9 \geq 0$. This occurs when $x \geq 4.5$. When $2x - 9 \geq 0$, our equation is $2x - 9 = 5$, so $x = 7$. Because 7 is greater than 4.5, our solution $x = 7$ satisfies the restrictions on x for this case.
- *Case 2:* $2x - 9 < 0$. This occurs when $x < 4.5$. When $2x - 9 < 0$, we have $|2x - 9| = -(2x - 9)$, so our equation is $-(2x - 9) = 5$. The solution to this equation is $x = 2$. Because $x = 2$ satisfies the inequality $2x - 9 < 0$, it satisfies the restrictions for this case. So, it is a valid solution.

Combining our two cases, we have $x = 2$ and $x = 7$ as our solutions.

Concept: We can often handle problems involving the absolute value of an expression by considering cases corresponding to variable values that make the expression negative or nonnegative.

Solution 3: What else must always be positive? Just like absolute value, squares must always be positive. So, in an expression like $|x^2|$ or $|x|^2$, the absolute value signs are redundant. We can write each as simply x^2. This gives us the idea of squaring the given equation:

$$|2x - 9|^2 = 5^2.$$

The absolute value sign is now redundant, so we have $(2x - 9)^2 = 5^2$, which rearranges as

$$4x^2 - 36x + 56 = 0.$$

Dividing by 4, then factoring, gives $(x - 2)(x - 7) = 0$, so our potential solutions are $x = 2$ and $x = 7$. Because we squared the equation as a step, we go back and check for extraneous solutions. We find that both solutions are valid.

Solution 4: Use the number line. Because $|2x - 9| = 5$, we know that $2x$ is 5 units from 9 on the number line. There are two numbers that are 5 units from 9 on the number line, $9 - 5 = 4$ and $9 + 5 = 14$. Therefore, $2x$ must equal 4 or 14. Solving $2x = 4$ gives $x = 2$ and solving $2x = 14$ gives $x = 7$.

See if you can also solve the problem by graphing the function $y = |2x - 9|$. □

What if we have two absolute value expressions in a function?

Problem 20.9: Let $f(x) = |x - 3| + |x + 2|$. Graph $y = f(x)$.

Solution for Problem 20.9: We graphed $y = |x|$ by considering separately the cases where the expression inside the absolute value is negative and where it is nonnegative. We try the same here. We have three cases to consider based on whether $x - 3$ and $x + 2$ are both positive, both negative, or one is positive and the other negative. We find the boundaries for the cases by locating values of x for which $x - 3 = 0$ or $x + 2 = 0$. These are $x = 3$ and $x = -2$.

- *Case 1: Both $x - 3$ and $x + 2$ are nonnegative.* This occurs when $x \geq 3$. Here, we have

  $$f(x) = |x - 3| + |x + 2| = x - 3 + x + 2 = 2x - 1.$$

  Another way to look at this is to think of $|x - 3|$ as the distance between x and 3 on the number line, and to think of $|x + 2| = |x - (-2)|$ as the distance between x and -2 on the number line. When $x \geq 3$, x is $x - 3$ more than 3 and $x - (-2) = x + 2$ more than -2. So, the sum of the distances from x to 3 and to -2 is $(x - 3) + (x + 2) = 2x - 1$. Therefore, if $x \geq 3$, we have $|x - 3| + |x + 2| = 2x - 1$, as before.

- *Case 2: Exactly one of $x - 3$ and $x + 2$ is negative.* This occurs when $-2 \leq x < 3$. (Make sure you see why.) In this range, $x - 3$ is negative but $x + 2$ is not, so

  $$f(x) = |x - 3| + |x + 2| = -(x - 3) + x + 2 = 5.$$

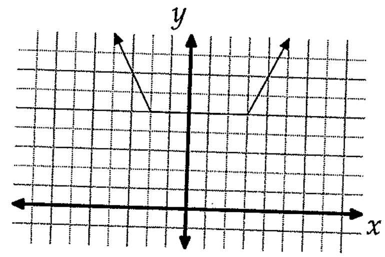
Once again, we can turn to the number line to see why $|x-3|+|x+2|=5$ when $-2 \le x < 3$. When x is between -2 and 3, the sum of the distances from x to -2 and to 3 simply equals the distance from -2 to 3, which is 5. Therefore, if $-2 \le x < 3$, we have $|x-3|+|x+2|=5$.

- *Case 3: Both $x-3$ and $x+2$ are negative.* This occurs when $x < -2$. Here, we have

$$f(x) = |x-3|+|x+2| = -(x-3)-(x+2) = -2x+1.$$

As with the first two cases, we could also have used the number line in this case.

We graph $y = f(x) = |x-3|+|x+2|$ by graphing each of these pieces. For $x \ge 3$, we graph $y = 2x-1$. For $-2 \le x < 3$, we graph $y = 5$. And for $x < -2$, we graph $y = -2x+1$. The result is shown at right.



One bit of interesting information we can read from our graph is that the smallest possible value of $f(x)$ is 5. This is not immediately obvious from looking at the equation $f(x) = |x-3|+|x+2|$. □

We can graph functions with two absolute value expressions, but how about solving equations with two absolute value expressions?

Problem 20.10: Find all solutions to the equation $|x+3|+|2-5x| = 7$.

Solution for Problem 20.10: We might try isolating an absolute value expression and squaring, but that's going to get nasty in a hurry. Instead, we use intervals, because they worked when graphing an expression similar to our left side. We first note that $x+3$ is nonnegative for $x \ge -3$ and $2-5x$ is nonnegative when $2-5x \ge 0$, or $x \le \frac{2}{5}$. We are now ready to set up our cases.

- *Case 1: $x > \frac{2}{5}$.* This makes $x+3$ nonnegative and $2-5x$ negative, so our equation is

$$x+3+[-(2-5x)] = 7.$$

The solution to this equation is $x = 1$, which satisfies $x > \frac{2}{5}$. So, it meets the restriction of this case.

- *Case 2: $-3 \le x \le \frac{2}{5}$.* This makes both $x+3$ and $2-5x$ nonnegative, so our equation is

$$x+3+2-5x = 7,$$

which gives us $x = -\frac{1}{2}$. This meets the restrictions of this case, so it is a valid solution.

- *Case 3: $x < -3$.* This makes $x+3$ negative and $2-5x$ nonnegative, so our equation is

$$-(x+3)+2-5x = 7,$$

which gives $x = -\frac{4}{3}$. This value *does not* meet the restriction of this case because $-\frac{4}{3}$ is not less than -3. Therefore, we cannot conclude that $x = -\frac{4}{3}$ is a valid solution to the original equation. (Plug $x = -\frac{4}{3}$ into the original equation and see what happens!)

Combining all three cases gives us $x = 1$ and $x = -\frac{1}{2}$ as our solutions.

WARNING!!	When you use casework to solve an absolute value equation, you must make sure the solutions you get in each case are among the permissible values for that case. One good way to test that you've done this is to test all your solutions in the original equation.

□

We've graphed functions that are absolute values of expressions, but what if x and y are both inside absolute value signs?

Problem 20.11: Graph the equation $|x+1|+|y-2|=5$.

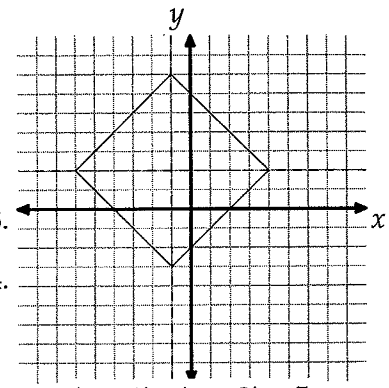
Solution for Problem 20.11: Casework has served us well so far with absolute value, so we'll stick with it. We consider x and y separately.

We have $|x+1| = x+1$ when $x \geq -1$ and $|x+1| = -x-1$ when $x < -1$.

We have $|y-2| = y-2$ when $y \geq 2$ and $|y-2| = -(y-2) = 2-y$ when $y < 2$.

Now we combine these to produce four cases:

- $x \geq -1, y \geq 2$. Our equation then is $x+1+y-2=5$, or $x+y=6$.
- $x \geq -1, y < 2$. Our equation then is $x+1+2-y=5$, or $x-y=2$.
- $x < -1, y \geq 2$. Our equation then is $-x-1+y-2=5$, or $-x+y=8$.
- $x < -1, y < 2$. Our equation then is $-x-1+2-y=5$, or $-x-y=4$.



$|x+1|+|y-2|=5$

We graph each of the pieces, careful not to go outside the boundaries of each case. In the graph at right, the dashed lines $y=2$ and $x=-1$ divide the plane into the four regions corresponding to our cases. □

Exercises

20.2.1 Solve the following two equations.

(a) $|r+3|-7=9$. (b) $|r+8|+7=4$.

20.2.2 If $|x-2|=p$, where $x<2$, then which of the following expressions must equal $x-p$?

(A) -2 (B) 2 (C) $2-2p$ (D) $2p-2$ (E) $|2p-2|$

(Source: AMC 10)

20.2.3 Solve the following two equations.

(a) $|6x-7|+3=12$ (b) $2|3-5x|=7$

20.2.4 Solve the equation $|y-6|+2y=9$.