Lesson 93 – Bayes' Theorem

HL2 Math - Santowski

Bayes' Theorem

- Main theorem:

Suppose we know $P(E \mid F)$

We would like to use this information to find $P(F \mid E)$ if possible.

Discovered by Reverend Thomas Bayes

Bayes' Theorem

- Shows the relation between one conditional probability and its inverse.

- Provides a mathematical rule for revising an estimate or forecast in light of experience and observation.

- Relates

-Prior Probability of A, P(A), is the probability of event A not concerning its associated event B

- Prior Probability of B, P(B), is the probability of B not concerning A

- Conditional Probability of B given A, $P(B|A)$. Also called the likelihood

-Conditional Probability of A given B, $P(A|B)$. Also called the posterior probability.

Bayes' Theorem

- Bayes' theorem is most commonly used to estimate the state of a hidden, causal variable ***H*** based on the measured state of an observable variable ***D***:

Likelihood

Prior

$$p(H \mid D) = \frac{p(D \mid H)p(H)}{p(D)}$$

Posterior

Evidence

Simple example of prior, conditional, and posterior probability

- A. Dice 1 lands on 3
- B. Dice 2 lands on 1
- C. The dice sum to 8

- The prior probability of A, P(A), is ….
- The prior probability of B, P(B), is ……
- The prior probability of C, P(C), is …….
- The conditional probability of event C given that A occurs, P(C│A), is ……
- The posterior probability, P(A│C), is ……

What is P(C│B)? Answer: 0

Simple example of prior, conditional, and posterior probability

A. Dice 1 lands on 3

B. Dice 2 lands on 1

C. The dice sum to 8

- The prior probability of A, P(A), is 1/6.
- The prior probability of B, P(B), is 1/6.
- The prior probability of C, P(C), is 5/36
- The conditional probability of event C given that A occurs, $P(C|A)$, is 1/6
- The posterior probability, $P(A|C)$, is 1/5

What is $P(C|B)$? Answer: 0

Example 1

- 1% of women at age forty who participate in routine screening have breast cancer.
- 80% of women with breast cancer will get positive mammographies.
- 9.6% of women without breast cancer will also get positive mammographies.

A woman in this age group had a positive mammography in a routine screening. What is the probability that she actually has breast cancer?

Without Bayes' Theorem

- Create a large sample size and use probabilities given in the problem to work out the problem.
- Assume, for example, that 10,000 women participate in a routine screening for breast cancer. 1%, or 100 women, have breast cancer. 80% of women with breast cancer, 80 women, will get positive mammographies. 9.6%,950 women, of the 9900 women who don't have breast cancer will also get positive mammographies.
- Create a table using the numbers obtained from the assumed sample size and determine the answer.

Without Bayes' Theorem cont.

	With breast cancer	**Without breast cancer**	**Total**
Positive Mammographies	80 women	950 women	1030 women
Negative Mammographies	20 women	8950 women	8970 women
Total	100 women	9900 women	10000 women

Out of the 1030 women who get positive mammographies only 80 actually have breast cancer, therefore, the probability is 80/1030 or 7.767%

Bayes' Theorem:

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

- The posterior probability is equal to the conditional probability of event B given A multiplied by the prior probability of A, all divided by the prior probability of B.

Using Bayes' Theorem

- 1% of women at age forty who participate in routine screening have breast cancer.

$P(B) = 0.01$

- 80% of women with breast cancer will get positive mammographies.

$P(+|B) = 0.8$

- 9.6% of women without breast cancer will also get positive mammographies.

$P(+|B') = 0.096$

- A woman in this age group had a positive mammography in a routine screening. What is the probability that she actually has breast cancer?

Find $P(B|+)$

Using Bayes' Theorem cont.

$$P(B|+) = \frac{P(+|B)\,P(B)}{P(+)}$$

- P(B), P(+|B), and P(+|B') are known. P(+) is needed to find P(B|+)
- $P(+) = P(+|B)\,P(B) + P(+|B')\,P(B')$

$$P(+) = (0.8)\,(0.01) + (0.096)\,(0.99)$$

$$P(+) = 0.1030$$

$$P(B|+) = \frac{(0.8)\,(0.01)}{(0.1030)}$$

$$P(B|+) = 0.07767$$

Example 2

- Two different suppliers, A and B, provide a manufacturer with the same part.
- All supplies of this part are kept in a large bin. in the past, 5% of the parts supplied by A and 9% of the parts supplied by B have been defective.
- A supplies four times as many parts as B

Suppose you reach into the bin and select a part, and find it is nondefective. What is the probability that it was supplied by A?

Solution

- 5% of the parts supplied by A and 9% of the parts supplied by B have been defective.

$P(D|A) = 0.95$ $P(D|B) = 0.91$

- A supplies four times as many parts as B

$P(A) = 0.8$ $P(B) = 0.2$

- Suppose you reach into the bin and select a part, and find it is nondefective. What is the probability that it was supplied by A?

Find $P(A|D)$

Solution cont.

$$P(A|D) = \frac{P(D|A)\,P(A)}{P(D)}$$

$P(D|A) = 0.95$ $\quad\quad$ $P(A) = 0.8$

$$P(D) = P(D|A)\,P(A) + P(D|B)\,P(B)$$

$$P(D) = (0.95)(0.8) + (0.91)(0.2) = 0.942$$

$$P(A|D) = \frac{(0.95)(0.8)}{(0.942)}$$

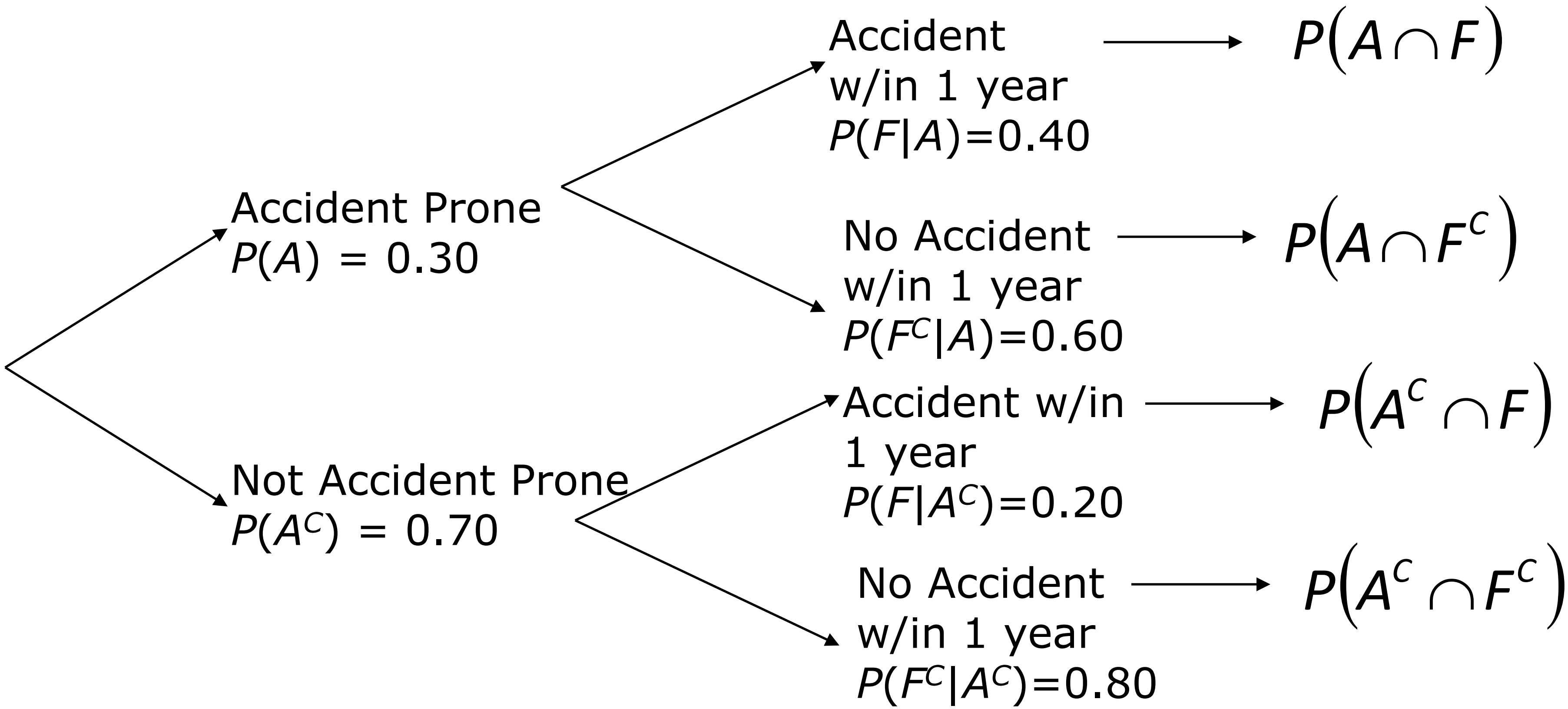
$$P(A|D) = 0.8068$$

Bayes' Theorem

- An insurance company divides its clients into two categories: those who are accident prone and those who are not. Statistics show there is a 40% chance an accident prone person will have an accident within 1 year whereas there is a 20% chance non-accident prone people will have an accident within the first year.
- If 30% of the population is accident prone, what is the probability that a new policyholder has an accident within 1 year?

Bayes' Theorem

- Using Tree Diagrams:



Example #1

- Suppose bowl B1 has 2 red and 4 blue coins
- Suppose bowl B2 has 1 red and 2 blue coins
- Suppose bowl B3 has 5 red and 4 blue coins

- Then suppose that the probabilities for selecting the bowls is not the same but are
- P(B1) = 1/3 and P(B2) = 1/6 and P(B3) = 1/2

- Compute:
- (i) the probability of drawing a red coin (ANS 4/9)
- (ii) Assuming a red coin was drawn, find the probability that it came from bowl 1. (ANS 2/8)

Example #2

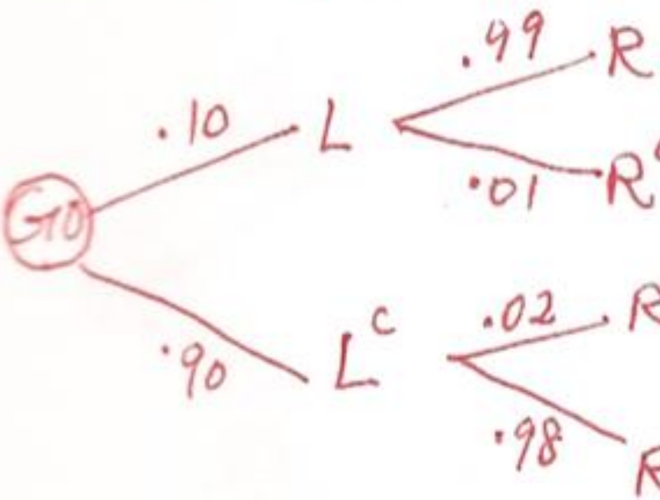
- Application: Approximately 1% of women aged 40-50 have breast cancer. A woman with breast cancer has a 91% chance of a positive test from a mammogram, while a without has a 11% chance of a false positive result. What is the probability that a woman has breast cancer given that she just had a positive test?

Example #3

1. A warning light is supposed to flash red in case engine oil pressure is low. The probability that the light is flashing given that the oil pressure is low is 0.99. Sometimes the light flashes when the oil pressure is not low and this occurs with probability 0.02. Suppose there is a 10% chance that the oil pressure really is low. What is the probability that the oil pressure is low if the light comes on?

1. A warning light is supposed to flash red in case engine oil pressure is low. The probability that the light is flashing given that the oil pressure is low is 0.99. Sometimes the light flashes when the oil pressure is not low and this occurs with probability 0.02. Suppose there is a 10% chance that the oil pressure really is low. What is the probability that the oil pressure is low if the light comes on?

R: red light
L: oil pressure low



① $P(LR) = (.10)(.99)$

② $P(LR^c) = (.10)(.01)$

③ $P(L^cR) = (.90)(.02)$

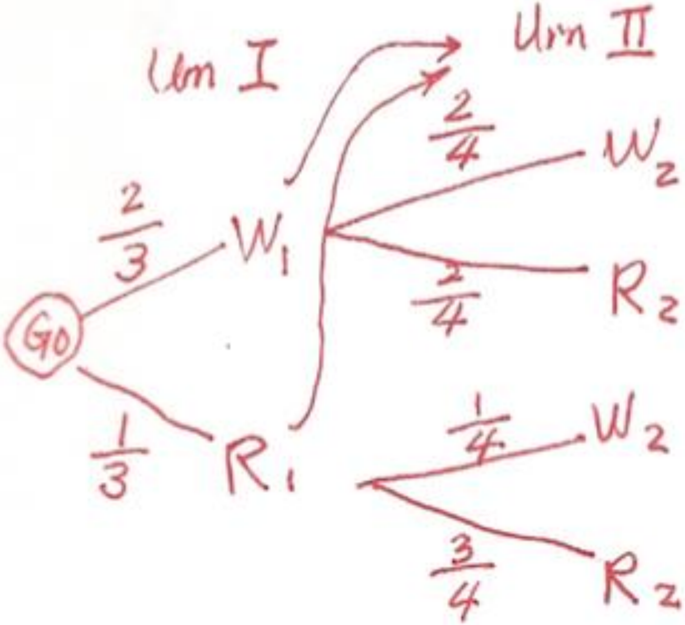
④ $P(L^cR^c) = (.90)(.98)$

Q: $P(L|R) = \dfrac{P(LR)}{P(R)} = \dfrac{①}{① + ③} = .85$

Example 4

2. Urn I contains 2 white chips and 1 red chip and Urn II contains 2 red chips and 1 white chip. A chip is selected at random from Urn I and transferred to Urn II. Then a chip is selected at random from Urn II. If the selected chip is red, what is the probability that the transferred chip was white?

2. Urn I contains 2 white chips and 1 red chip and Urn II contains 2 red chips and 1 white chip. A chip is selected at random from Urn I and transferred to Urn II. Then a chip is selected at random from Urn II. If the selected chip is red, what is the probability that the transferred chip was white?



① $P(W_1W_2) = \frac{2}{3} \times \frac{2}{4} = \frac{1}{3}$

② $P(W_1R_2) = \frac{2}{3} \times \frac{2}{4} = \frac{1}{3}$

③ $P(R_1W_2) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$

④ $P(R_1R_2) = \frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$

Q: $P(W_1|R_2) = \dfrac{P(W_1R_2)}{P(R_2)} = \dfrac{②}{② + ④} = \dfrac{4}{7}$

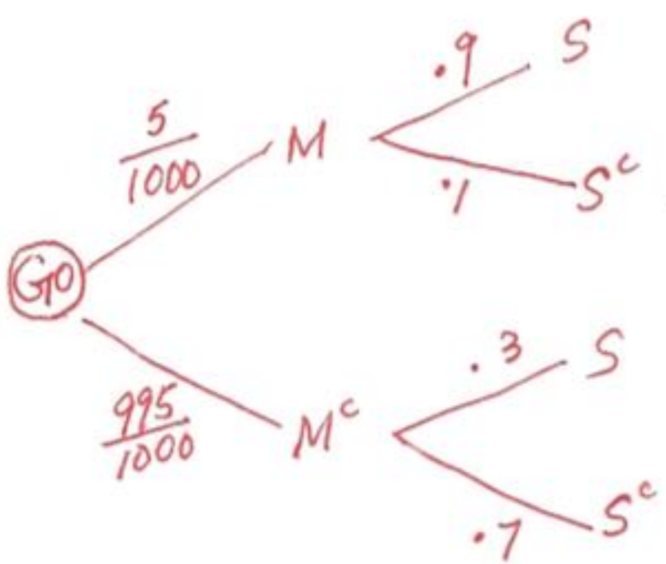
Example #5

3. At a certain university, 0.5% of students going to the infirmary seeking treatment are eventually diagnosed as having mononucleosis. Of those that do have mono, 90% complaining of a sore throat. But 30% of those not having mono also have sore throats. If a student comes to the infirmary and says she has a sore throat, what is the probability that she has mono?

3. At a certain university, 0.5% of students going to the infirmary seeking treatment are eventually diagnosed as having mononucleosis. Of those that do have mono, 90% complaining of a sore throat. But 30% of those not having mono also have sore throats. If a student comes to the infirmary and says she has a sore throat, what is the probability that she has mono?

M: Mono

S: Sore throat



① $P(MS) = \frac{5}{1000}(.9)$

② $P(MS^c) = \frac{5}{1000}(.1)$

③ $P(M^cS) = \frac{995}{1000}(.3)$

④ $P(M^cS^c) = \frac{995}{1000}(.7)$

Q: $P(M|S) = \frac{P(MS)}{P(S)} = \frac{①}{① + ③} \approx .015$

Example #6

3. Suppose that Martin, a student, knows 80% of the material to be covered on an upcoming True-False test. We take this to mean that on any randomly chosen test question there is a .8 probability that he will know the answer and we assume that in such a case he will give the correct answer. Also, we assume that if he does not know the answer to a particular question, he will guess and will have a .5 probability of getting it right. Suppose that a question is chosen at random.

   a. What is the probability that Martin will give the correct answer to the question?

3. Suppose that Martin, a student, knows 80% of the material to be covered on an upcoming True-False test. We take this to mean that on any randomly chosen test question there is a .8 probability that he will know the answer and we assume that in such a case he will give the correct answer. Also, we assume that if he does not know the answer to a particular question, he will guess and will have a .5 probability of getting it right. Suppose that a question is chosen at random.

   a. What is the probability that Martin will give the correct answer to the question?

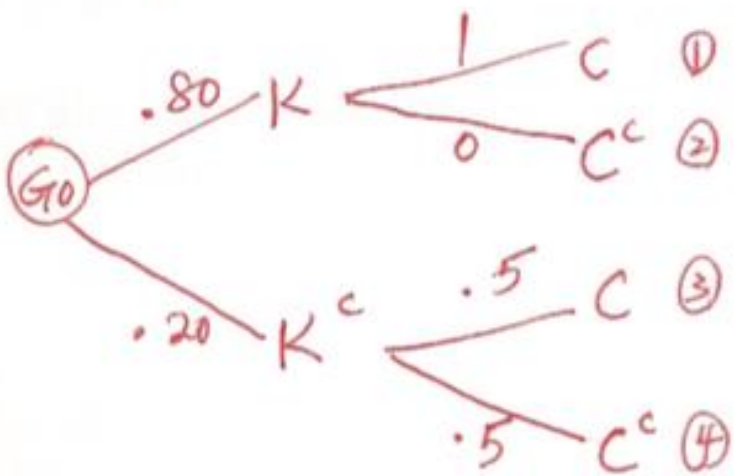
K: know

C: correct

$P(C) = ① + ③$

$= (.8)(1) + (.2)(.5)$

$= .9$



Example #6b

3. Suppose that Martin, a student, knows 80% of the material to be covered on an upcoming True-False test. We take this to mean that on any randomly chosen test question there is a .8 probability that he will know the answer and we assume that in such a case he will give the correct answer. Also, we assume that if he does not know the answer to a particular question, he will guess and will have a .5 probability of getting it right. Suppose that a question is chosen at random.

b. What is the probability that Martin knew the answer to the question, given that he gave the correct answer?

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$$P(K|C) = \frac{P(KC)}{P(C)} = \frac{①}{① + ③} = \frac{.8}{.9} = \frac{8}{9}$$

Example #7

- An employer is looking to fill some positions and several college graduates are interviewed. From past experience, we know that the employer will offer second interviews to 65% of the college graduates. Of those offered second interviews, 70% of them will be hired. Only 5% of college graduates not offered a second interview will be hired.

- Set up a tree diagram

Example #7 - CONTINUED

- (1) What is the probability of getting offered a second interview and not getting hired?
- (2) What is the probability of getting hired and not being offered a 2^{nd} interview
- (3) What is the probability of a randomly selected graduate being hired?
- (4) What is the probability of getting hired, given that a second interview was offered?
- (5) What is the probability of having had a second interview, given that someone was hired

Example #8

- At an assembly plant, 14% of the gadgets that come off the line are defective. Quality control checks the gadgets and 96% of the defective gadgets are correctly discarded. However, there is a 9% chance that a working gadget is mistakenly discarded. One gadget is selected at random.
- Draw a tree diagram

Example #8 - CONTINUED

- (1) What is the probability that the gadget is not defective?
- (2) What is the probability that the gadget is defective and has been discarded?
- (3) Given that the gadget is defective, what is the probability that it will not be discarded?
- (4) Given that the gadget is not discarded, what is the probability that it is defective?
- (5) What is the probability that a gadget is not discarded?