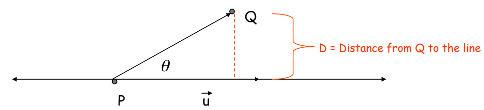
LESSON 90 – DISTANCES IN 3 SPACE

HL2 - Santowski

Distance Between a Point and a Line

In the picture below, Q is a point not on the line , P is a point on the line, $\vec{u}$ is a direction vector for the line and θ is the angle between $\vec{u}$ and $\vec{PQ}$.



Obviously, $\sin\theta = \dfrac{D}{\|\vec{PQ}\|}$ or $D = \|\vec{PQ}\|\sin\theta$

We know from Section 7.4 on cross products that

$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$.

Thus, $\|\vec{PQ}\times\vec{u}\| = \|\vec{PQ}\|\|\vec{u}\|\sin\theta$

or dividing both sides by $\|\vec{u}\|$

$$\frac{\|\vec{PQ}\times\vec{u}\|}{\|\vec{u}\|} = \|\vec{PQ}\|\sin\theta$$

So if, $D = \|\vec{PQ}\|\sin\theta$ then from above, $D = \dfrac{\|\vec{PQ}\times\vec{u}\|}{\|\vec{u}\|}$.

Distance Between a Point and a Line

The distance, D, between a line and a point Q not on the line is given by

$$D = \frac{\|\vec{PQ}\times\vec{u}\|}{\|\vec{u}\|}$$

where $\vec{u}$ is the direction vector of the line and P is a point on the line.

Example 1: Find the distance between the point Q (1, 3, -2) and the line given by the parametric equations:

$$x = 2 + t,\quad y = -1 - t \quad\text{and}\quad z = 3 + 2t$$

Solution: From the parametric equations we know the direction vector, $\vec{u}$ is ‹ 1, -1, 2 › and if we let t = 0, a point P on the line is P (2, -1, 3).

Thus $\vec{PQ}$ = ‹ 2-1, -1-3, 3-(-2) › = ‹ 1, -4, 5 ›

Find the cross product:
$$\vec{PQ}\times\vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -4 & 5 \\ 1 & -1 & 2 \end{vmatrix} = -3\vec{i} + 3\vec{j} + 3\vec{k}$$

Using the distance formula:

$$D = \frac{\|\vec{PQ}\times\vec{u}\|}{\|\vec{u}\|} = \frac{\sqrt{(-3)^2 + 3^2 + 3^2}}{\sqrt{1^2 + (-1)^2 + 2^2}} = \frac{\sqrt{27}}{\sqrt{6}} = \sqrt{\frac{9}{2}} \approx 2.12$$

DISTANCES BETWEEN LINES

- Example #2:
- (a) Show that the lines are skew.
  L1: x = 1 + t y = –2 + 3t z = 4 – t
  L2: x = 2s y = 3 + s z = –3 + 4s
- (b) Find the distance between them.

DISTANCES BETWEEN LINES

- Since the two lines L_1 and L_2 are skew, they can be viewed as lying on two parallel planes P_1 and P_2.
  - The distance between L_1 and L_2 is the same as the distance between P_1 and P_2.

DISTANCES BETWEEN LINES

- The common normal vector to both planes must be orthogonal to both
- $v_1 = \langle 1, 3, -1 \rangle$ (direction of L_1)
- $v_2 = \langle 2, 1, 4 \rangle$ (direction of L_2)

DISTANCES BETWEEN LINES

- If we put $s = 0$ in the equations of L_2, we get the point $(0, 3, -3)$ on L_2.
  - So, an equation for P_2 is:

$$13(x - 0) - 6(y - 3) - 5(z + 3) = 0$$

or $$13x - 6y - 5z + 3 = 0$$

DISTANCES BETWEEN LINES

- If we now set $t = 0$ in the equations for L_1, we get the point $(1, -2, 4)$ on P_1.

DISTANCES BETWEEN LINES

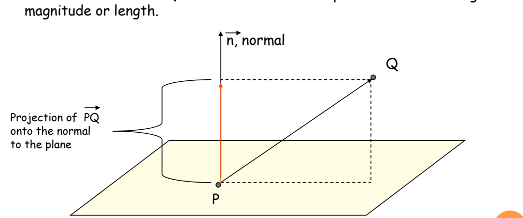
- So, the distance between L_1 and L_2 is the same as the distance from $(1, -2, 4)$ to $13x - 6y - 5z + 3 = 0$.

$$d = \frac{8}{\sqrt{230}} = 0.53$$

<u>Distance Between a Point and a Plane</u>

Let P be a point in the plane and let Q be a point not in the plane. We are interested in finding the distance from the point Q to the plane that contains the point P.

We can find the distance between the point, Q, and the plane by projecting the vector from P to Q onto the normal to the plane and then finding its magnitude or length.



Thus the distance from Q to the plane is the length or the magnitude of the projection of the vector $\vec{PQ}$ onto the normal.

Distance Between a Point and a Plane

If the distance from Q to the plane is the length or the magnitude of the projection of the vector PQ onto the normal, we can write that mathematically:

$$\text{Distance from Q to the plane} = \left\| proj_{\vec{n}} \overrightarrow{PQ} \right\|$$

Now, recall from section 7.3, $proj_{\vec{n}} \overrightarrow{PQ} = \left(\dfrac{\overrightarrow{PQ} \cdot \vec{n}}{\|\vec{n}\|^2} \right) \cdot \vec{n}$

So taking the magnitude of this vector, we get:

$$\left\| proj_{\vec{n}} \overrightarrow{PQ} \right\| = \left\| \left(\frac{\overrightarrow{PQ} \cdot \vec{n}}{\|\vec{n}\|^2} \right) \cdot \vec{n} \right\| = \frac{\left|\overrightarrow{PQ} \cdot \vec{n}\right|}{\|\vec{n}\|^2} \cdot \|\vec{n}\| = \frac{\left|\overrightarrow{PQ} \cdot \vec{n}\right|}{\|\vec{n}\|}$$

Distance Between a Point and a Plane

The distance from a plane containing the point P to a point Q not in the plane is

$$D = \left\| proj_{\vec{n}} \overrightarrow{PQ} \right\| = \frac{\left|\overrightarrow{PQ} \cdot \vec{n}\right|}{\|\vec{n}\|}$$

where n is a normal to the plane.

Example 1: Find the distance between the point Q (3, 1, -5) to the plane 4x + 2y - z = 8.

Solution: We know the normal to the plane is ‹4, 2, -1› from the general form of a plane. We can find a point in the plane simply by letting x and y equal 0 and solving for z: P (0, 0, -8) is a point in the plane.

Thus the vector, $\overrightarrow{PQ}$ = ‹3-0, 1-0, -5-(-8)› = ‹3, 1, 3›

Now that we have the vector $\overrightarrow{PQ}$ and the normal, we simply use the formula for the distance between a point and a plane.

$$D = \left\| proj_{\vec{n}} \overrightarrow{PQ} \right\| = \frac{\left|\overrightarrow{PQ} \cdot \vec{n}\right|}{\|\vec{n}\|} = \frac{\left|\langle 3,1,3\rangle \cdot \langle 4,2,-1\rangle\right|}{\sqrt{4^2 + 2^2 + (-1)^2}}$$

$$D = \frac{|12 + 2 - 3|}{\sqrt{16 + 4 + 1}} = \frac{11}{\sqrt{21}} \approx 2.4$$

Example 2: Find the distance between the point (1, 2, 3) and line $2x - y + z = 4$

Solution:

Distances Between Points & Planes

- Find the distance between the parallel planes

  10x + 2y – 2z = 5 and 5x + y – z = 1

Distances Between Points & Planes

- First, we note that the planes are parallel because their normal vectors are parallel.

  i.e. <10, 2, –2> and <5, 1, –1>

Distances Between Points & Planes

- To find the distance D between the planes, we choose any point on one plane and calculate its distance to the other plane.
  - In particular, if we put y = z =0 in the equation of the first plane, we get 10x = 5.
  - So, (½, 0, 0) is a point in this plane.

Distances Between Points & Planes

So, the distance between the planes is $\sqrt{3}/6$