Lesson 68 – Infinite Series – Integral Test

IBHL - Calculus - Santowski

Lesson Objectives

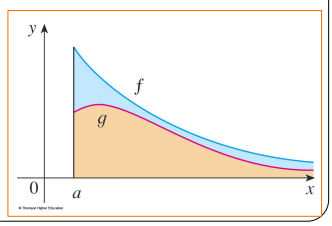
- Introduce the Integral Test for predicting convergences and divergences of infinite series
- Practice using the Integral Test
- Introduce the idea of the *p*-series

COMPARISON THEOREM

- Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.
  - If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.
  - If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

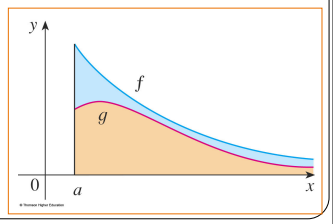
COMPARISON THEOREM

- We omit the proof of the theorem.
- However, the figure makes it seem plausible.



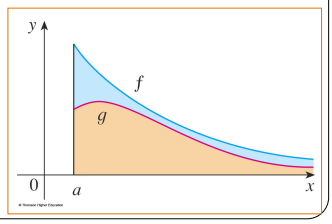
COMPARISON THEOREM

- If the area under the top curve y = f(x) is finite, so is the area under the bottom curve y = g(x).

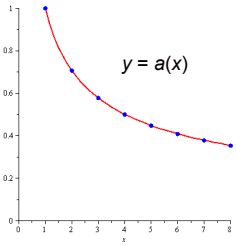


COMPARISON THEOREM

- If the area under y = g(x) is infinite, so is the area under y = f(x).



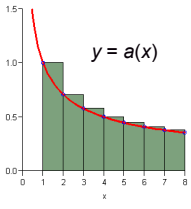
The Integral Test



Suppose that we have a sequence $\{a_k\}$ and we associate it with a continuous function $y = a(x)$, as we did a few lessons ago. . .

Now we add some enlightening pieces to our diagram….

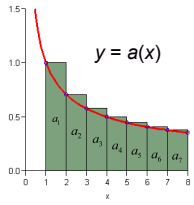
The Integral Test



Suppose that we have a sequence $\{a_k\}$ and we associate it with a continuous function $y = a(x)$, as we did a few lessons ago. . .

Look at the graph. . .
What do you see?

The Integral Test

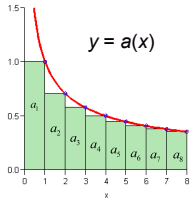


So $\int_1^\infty a(x)dx \le \sum_{n=1}^\infty a_n$

If the integral converges / diverges

so does the series.

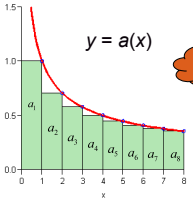
The Integral Test



Suppose that we have a sequence $\{a_k\}$ and we associate it with a continuous function $y = a(x)$, as we did a few lessons ago. . .

Now look at this graph. . .
What do you see?

The Integral Test



So $\sum_{n=2}^\infty a_n \le \int_1^\infty a(x)dx$

Why 2?

If the integral converges / diverges

so does the series.

The Integral Test

The Integral Test:

Suppose for all $x \ge 1$, the function $a(x)$ is **continuous**, **positive**, and **decreasing**.

Consider the series $\sum_{n=1}^\infty a_n$ and the integral $\int_1^\infty a(x)dx$

If the integral **converges**, then so does the series.
If the integral **diverges**, then so does the series.

REMARK:

Also, it is not necessary that f(x) be always decreasing. What is important is that f(x) be *ultimately* decreasing, that is, decreasing for larger than some number N.

Examples

- Use the Integral Test to determine whether the series below are convergent or divergent

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^4}$

(c) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ (d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$

(e) $\sum_{n=1}^{\infty} ne^{-n}$ (f) $\sum_{n=1}^{\infty} \frac{n+2}{n+1}$

(g) $\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$ (h) $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$

Exercises 10.3

Applying the Integral Test

Use the Integral Test to determine if the series in Exercises 1–10 converge or diverge. Be sure to check that the conditions of the Integral Test are satisfied.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$
2. $\sum_{n=1}^{\infty} \frac{1}{n^{0.2}}$
3. $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$
4. $\sum_{n=1}^{\infty} \frac{1}{n+4}$
5. $\sum_{n=1}^{\infty} e^{-2n}$
6. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
7. $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$
8. $\sum_{n=2}^{\infty} \frac{\ln(n^2)}{n}$
9. $\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$
10. $\sum_{n=2}^{\infty} \frac{n-4}{n^2-2n+1}$

The *p*-series

- Use the Integral Test to determine whether the following series converge or diverge?

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^4}$

(c) $\sum_{n=1}^{\infty} \frac{1}{n}$ (d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(e) $\sum_{n=1}^{\infty} \sqrt{n^{-3}}$ (f) $\sum_{n=1}^{\infty} \sqrt[4]{n^{-12}}$

SUMMARY

(g) $\sum_{n=1}^{\infty} \frac{1}{n^p}$