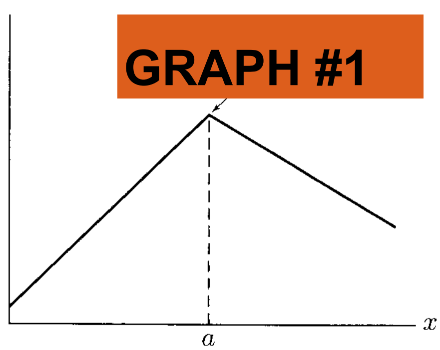
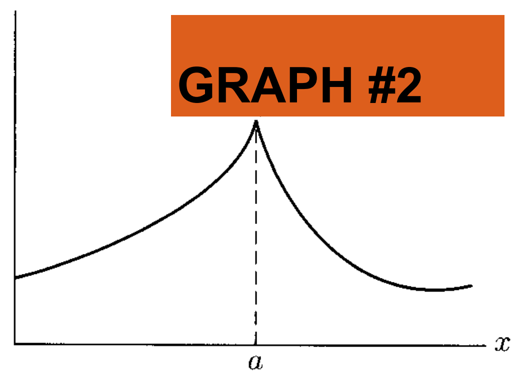
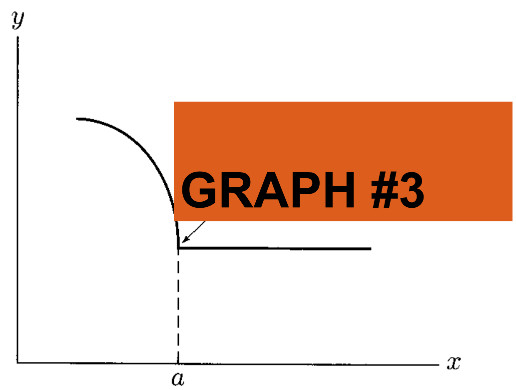
Lesson 61 – Continuity and Differentiability

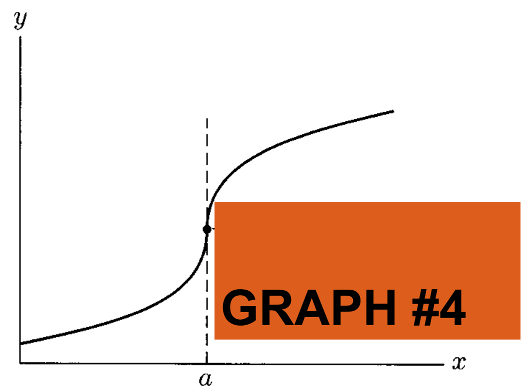
HL Math - Santowski

Draw the graphs of the derivatives of the following graphs:









Lesson Objectives

- Objectives:

- Be able to make a connection between a differentiablity and continuity.

- Be able to use the alternative form of the derivative to determine if the derivative exists at a specific point.

(A) Continuity

- We can introduce another characteristic of functions ➔ that of continuity. We can understand continuity in several ways:

- (1) a continuous process is one that takes place gradually, smoothly, without interruptions or abrupt changes

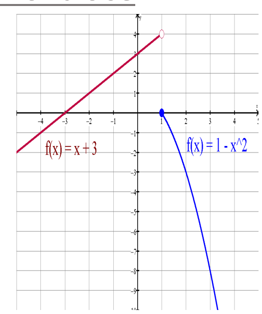
- (2) a function is continuous if you can take your pencil and can trace over the graph with one uninterrupted motion

(B) Conditions for Continuity

- a fcn is continuous at a given number, $x = a$, if:
- (i) $f(a)$ exists;
- (ii) $\lim_{x \to a} f(x)$ exists
- (iii) $f(a) = \lim_{x \to a} f(x)$
- In other words, if I can evaluate a function at a given value of $x = a$ and if I can determine the value of the limit of the function at $x = a$ and if we notice that the function value is the same as the limit value, then the function is continuous at that point.
- So a function is continuous over its domain if it is continuous at each point in its domain.

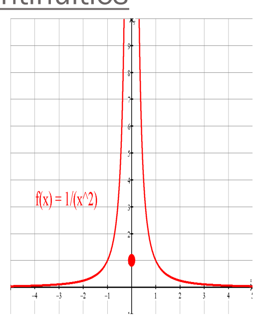
(C) Types of Discontinuities

- *(I) Jump Discontinuities:*
- ex $f(x) = \begin{cases} x+3, x<1 \\ 1-x^2, x \geq 1 \end{cases}$
- and it's limit and function values at $x = 1$.
- We notice our function values and our limits (LHL and RHL) "jump" from 4 to 0



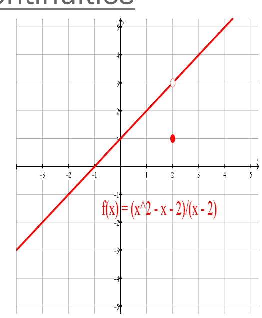
(C) Types of Discontinuities

- *(II) Infinite Discontinuities*
- ex. $f(x) = \begin{cases} \dfrac{1}{x^2} & ; x \neq 0 \\ 1; x = 0 \end{cases}$
- and it's limit and function values at $x = 0$.
- The left hand limit and right hand limits are both infinite although the function value is 1



(C) Types of Discontinuities

- *(III) Removable Discontinuities*
- Ex $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x-2}; x \neq 2 \\ 1; x = 2 \end{cases}$
- and it's limit and function values at $x = 2$.
- The left hand limit and right hand limits are equal to 3 although the function value is 1



(D) Examples

- Find all numbers, $x = a$, for which each function is discontinuous. For each discontinuity, state which of the three conditions are not satisfied.
- (i) $f(x) = \dfrac{x}{(x+1)^2}$ (ii) $g(x) = \dfrac{x^2 - 9}{x - 3}$
- (iii) $g(x) = \begin{cases} 2x^4 - 3x^3 - x^2 + x - 1; x \leq 2 \\ \dfrac{x^2 + 2x + 3}{x - 1}; x > 2 \end{cases}$
- (iv) $h(x) = \begin{cases} \dfrac{x^2 + 3x - 10}{x - 2}; x \neq 2 \\ 7; x = 2 \end{cases}$

(E) Continuity and Differentiability – An Algebraic Perspective

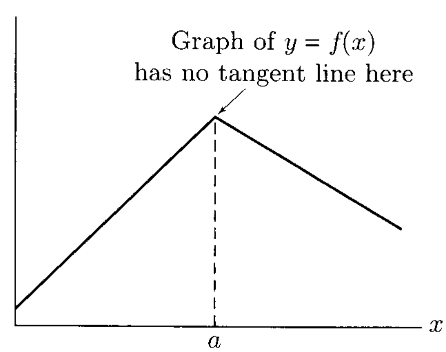
- In a previous lesson, we defined differentiability of f(x) at x = a in terms of a limit. Recall that if f(x) is differentiable at x = a, we can evaluate the following limit to determine f'(a).

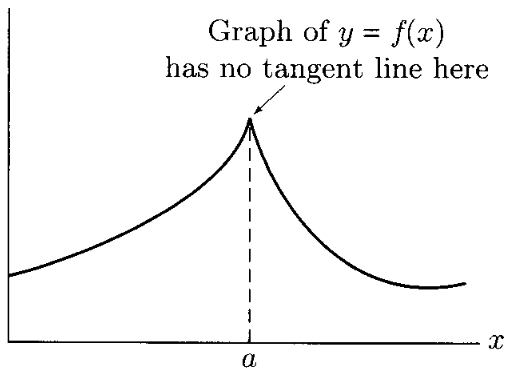
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

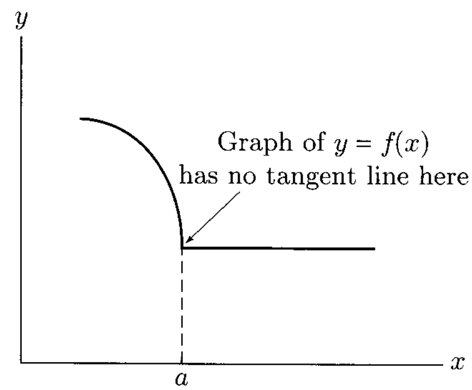
- Conversely, if this limit does not exist, then f(x) is nondifferentiable at x = a.

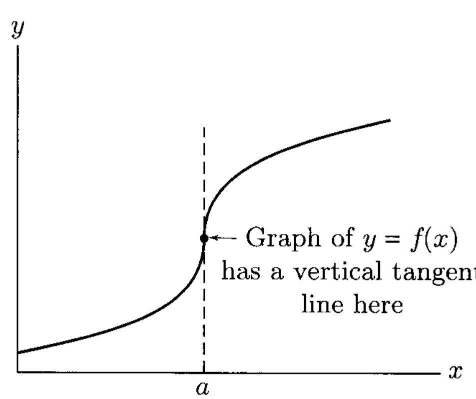
(E) Continuity and Differentiability

- Recall the fundamental idea that a derivative at a point is really the idea of a limiting sequence of secant slopes (or tangent line) drawn to a curve at a given point
- Now, if a function is continuous at a given point, from this fixed point, try drawing secant lines from the left side and secant lines from the right side and then try drawing a specific tangent slope at this point in the following diagrams
- Conclusion ➔ you can only differentiate a function where is it is derivative is continuous



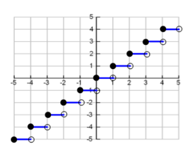






I. Differentiability and Continuity

If a function is NOT continuous at a certain point, (say, x = c) then it is also not differentiable at x =c.



Greatest Integer Function: f(x)=[[x]]

Let's look at when x = 0

We notice the graph is not continuous at x = 0 because we have a gap.

We can't take a derivative at a gap

We can show this algebraically by using an alternative form of the limit definition of the derivative.

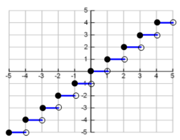
$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

This requires that the one-sided limits exist and are equal

$$\lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c}$$

I. Differentiability and Continuity

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$



$$\lim_{x \to 0^-} \frac{[[x]] - 0}{x - 0} \longrightarrow \lim_{x \to 0^-} \frac{[[x]]}{x} \longrightarrow \infty$$

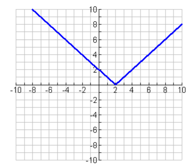
$$\lim_{x \to 0^+} \frac{[[x]] - 0}{x - 0} \longrightarrow \lim_{x \to 0^+} \frac{[[x]]}{x} \longrightarrow 0$$

x	-.5	-.1	-.01	0	.01	.1	.5
f(x)	-1/-.5	-1/-.1	-1/-.01	?	0/.01	0/.1	0/.5

I. Differentiability and Continuity

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

Example: A graph that contains a sharp turn



$f(x) = |x - 2|$

$$f'(c) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

$$\lim_{x \to 2^-} \frac{|x - 2| - f(2)}{x - 2} \longrightarrow \lim_{x \to 2^-} \frac{|x - 2|}{x - 2} \longrightarrow -1$$

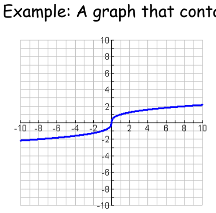
$$\lim_{x \to 2^+} \frac{|x - 2| - f(2)}{x - 2} \longrightarrow \lim_{x \to 2^+} \frac{|x - 2|}{x - 2} \longrightarrow 1$$

Since the limits are not equal, we can conclude that the function is not differentiable at x = 2 and no tangent line exists at (2,0).

I. Differentiability and Continuity

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

Example: A graph that contains a Vertical Tangent Line



$f(x) = \sqrt[3]{x}$

$$f'(c) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$\lim_{x \to 0^-} \frac{\sqrt[3]{x} - f(0)}{x - 0} \longrightarrow \lim_{x \to 0^-} \frac{\sqrt[3]{x}}{x} \longrightarrow \infty$$

$$\lim_{x \to 0^+} \frac{\sqrt[3]{x} - f(0)}{x - 0} \longrightarrow \lim_{x \to 0^+} \frac{\sqrt[3]{x}}{x} \longrightarrow \infty$$

Since the limit is infinite, we can conclude that the tangent line is vertical at x = 0.

I. Differentiability and Continuity

Big Ideas

1. If a function is differentiable (you can take the derivative) at x = c, then it is continuous at x = c. So, differentiability implies continuity.
2. It is possible for a function to be continuous at x = c and NOT be differentiable at x =c. So, continuity does not imply differentiability (Sharp turns in graphs and vertical tangents).

Example #1

- Is the given function y = f(x) (as given below) continuous and differentiable at x = 2?

$$f(x) = \begin{cases} \dfrac{x^2 - 6x + 8}{x - 2} & \text{if } x \neq 2 \\ 3 & \text{if } x \neq 2 \end{cases}$$

Example #1 - SOLN

(1) Suppose we want to determine whether the function

$$f(x) = \begin{cases} \frac{x^2-6x+8}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$$

is differentiable at $x = 2$. You would first make sure that it is continuous at $x = 2$: since

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{(x-4)(x-2)}{x-2} = 2 - 4 = -2$$

and $-2 \neq f(2) = 3$, $f(x)$ is not continuous at $x = 2$, so it cannot be differentiable at $x = 2$.

Example #2

- Now, let's change the 3 to a -2. Is the given function y = f(x) (as given below) now continuous and differentiable at x = 2?

$$f(x) = \begin{cases} \dfrac{x^2 - 6x + 8}{x - 2} & \text{if } x \neq 2 \\ -2 & \text{if } x \neq 2 \end{cases}$$

Example #2 - SOLN

(2) Suppose I changed the 3 to a -2:

$$f(x) = \begin{cases} \frac{x^2-6x+8}{x-2} & \text{if } x \neq 2 \\ -2 & \text{if } x = 2 \end{cases}$$

and I wanted to know whether it was differentiable at $x = 2$. Well now $f(x)$ is continuous, so we can move on to differentiablity. There are two ways to see $f(x)$ is differentiable. First, notice that $f(x)$ is just the line $x - 4$, since we can rewrite $f(x)$ as

$$f(x) = \begin{cases} x - 4 & \text{if } x \neq 2 \\ -2 & \text{if } x = 2 \end{cases},$$

so all we did is remove the point $(2, -2)$ in the line $y = x - 4$ and then fill it in again. The other way would be to show

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

exists. Using this rewritten form of $f(x)$ for the limit is easier, and I'll leave it to you to check.

(F) Examples from AP

443 (AP). Suppose f is a function for which $\lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = 0$. Which of the following must be true, might be true, or can never be true?

a) $f'(2) = 2$

b) $f(2) = 0$

c) $\lim_{x\to 2} f(x) = f(2)$

d) $f(x)$ is continuous at $x = 0$.

e) $f(x)$ is continuous at $x = 2$.

(F) Examples from AP

444 (AP). For some nonzero real number a, define the function f as $f(x) = \begin{cases} \frac{x^2-a^2}{x-a} & x \neq a \\ 0 & x = a. \end{cases}$

a) Is f defined at a?

b) Does $\lim_{x\to a} f(x)$ exist? Justify your answer.

c) Is f continuous at a? Justify your answer.

d) Is f differentiable at a? Justify your answer.

(F) Examples from AP

445. If $\lim_{x\to a} f(x) = L$, which of the following statements, if any, *must* be true? Justify your answers.

a) f is defined at a.

b) $f(a) = L$.

c) f is continuous at a.

d) f is differentiable at a.

(F) Examples from AP

446. Let $f(x) = \begin{cases} ax & x \leq 1 \\ bx^2 + x + 1 & x > 1. \end{cases}$

a) Find all choices of a and b such that f is continuous at $x = 1$.

b) Draw the graph of f when $a = 1$ and $b = -1$.

c) Find the values of a and b such that f is differentiable at $x = 1$.

d) Draw the graph of f for the values of a and b found in part (c).

(F) Examples #7 & #8

1. Find the number c that makes

$$f(x) = \begin{cases} \dfrac{x-c}{c+1}, & \text{if } x \le 0 \\ x^2 + c, & \text{if } x > 0 \end{cases}$$

continuous for every x.

2. Find the values of a and b so that

$$f(x) = \begin{cases} ax + b, & \text{if } x < 0 \\ 2\sin(x) + 3\cos(x) & \text{if } x \ge 0 \end{cases}$$

is differentiable at $x = 0$.

1. Find the number c that makes

$$f(x) = \begin{cases} \dfrac{x-c}{c+1}, & \text{if } x \le 0 \\ x^2 + c, & \text{if } x > 0 \end{cases}$$

continuous for every x.

Solution:

Note that $f(x)$ is continuous for every $x \neq 0$.

$f(0) = \dfrac{0-c}{c+1} = \dfrac{-c}{c+1}$.

$\lim_{x\to 0^+} f(x) = 0^2 + c = c$.

$\lim_{x\to 0^-} f(x) = \dfrac{-c}{c+1}$.

Since $f(x)$ is continuous for every x, hence continuous for $x = 0$.

$\Rightarrow f(0) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$.

$\Rightarrow \dfrac{-c}{c+1} = c$.

$\Rightarrow c = 0$ or $c = -2$.

2. Find the values of a and b so that

$$f(x) = \begin{cases} ax + b, & \text{if } x < 0 \\ 2\sin(x) + 3\cos(x) & \text{if } x \ge 0 \end{cases}$$

is differentiable at $x = 0$.

Solution:

First of all, $f(x)$ must be continuous at $x = 0$. Hence $\lim_{x\to 0^-} f(x) = f(0)$.

$\Rightarrow b = 2\sin 0 + 3\cos 0 = 3$.

Second, find $f'(x)$:

$$f'(x) = \begin{cases} a, & \text{if } x < 0 \\ 2\cos(x) - 3\sin(x) & \text{if } x \ge 0 \end{cases}.$$

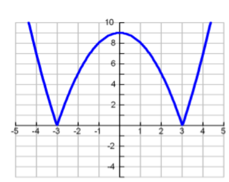
Since $f(x)$ is differentiable at $x = 0$. $\lim_{x\to 0^-} f'(x) = f'(0)$.

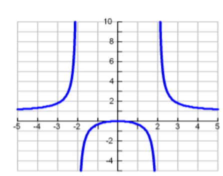
$\Rightarrow a = 2\cos 0 - 3\sin 0 = 2$.

Therefore $a = 2$, $b = 3$.

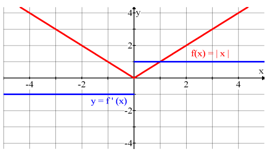
Example #9 - Graphic

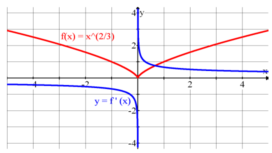
- Describe the x-values at which y = f(x) is differentiable?

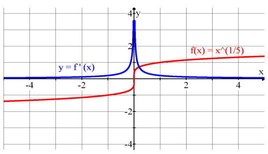




(F) Continuity and Differentiability – Examples







- Continuous functions are non-differentiable under the following conditions:
  - The fcn has a "corner" (ex 1)
  - The fcn has a "cusp" (ex 2)
  - The fcn has a vertical tangent (ex 3)
- This non-differentiability can be seen in that the graph of the derivative has a discontinuity in it!