Lesson 56 – Separable Differential Equations

Calculus - Santowski

Lesson Objectives

- 1. Solve separable differential equations with and without initial conditions
- 2. Solve problems involving exponential decay in a variety of application (Radioactivity, Air resistance is proportional to velocity, Continuously compounding interest, Population growth)

(A) Separable Equations

- So far, we have seen differential equations that can be solved by integration since our functions were relatively easy functions in one variable
- Ex. *dy*/*dx* = sin*x* - 1/*x*
- Ex. *dv*/*dt* = -9.8

(A) Separable Equations

- In Ex 1, we simply evaluated the indefinite integral of both sides

$$\frac{dy}{dx} = \sin x - \frac{1}{x}$$
$$\int\left(\frac{dy}{dx}\right)dx = \int\left(\sin x - \frac{1}{x}\right)dx$$
$$y = -\cos x - \ln x + C$$

- But what about the equation dy/dx = -x/y?
- If we tried finding antiderivatives or indefinite integrals ….

$$\frac{dy}{dx} = -\frac{x}{y}$$
$$\int\left(\frac{dy}{dx}\right)dx = \int\left(-\frac{x}{y}\right)dx$$
$$y = ???$$

SEPARABLE EQUATION

- A separable equation is a first-order differential equation in which the expression for *dy*/*dx* can be factored as a function of *x* times a function of *y*.
  - In other words, it can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

SEPARABLE EQUATIONS

- The name separable comes from the fact that the expression on the right side can be "separated" into a function of *x* and a function of *y*.

SEPARABLE EQUATIONS

- Equivalently, if $f(y) \neq 0$, we could write

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

where $h(y) = 1/f(y)$

SEPARABLE EQUATIONS

- To solve this equation, we rewrite it in the differential form $h(y)\,dy = g(x)\,dx$ so that:
  - All y's are on one side of the equation.
  - All x's are on the other side.

SEPARABLE EQUATIONS

- Then, we integrate both sides of the equation:

$$\int h(y)\,dy = \int g(x)\,dx$$

SEPARABLE EQUATIONS – Example #1

a. Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

b. Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

SEPARABLE EQUATIONS – Example #1 - SOLN

- We write the equation in terms of differentials and integrate both sides:
  - $y^2\,dy = x^2\,dx$
  - $\int y^2\,dy = \int x^2\,dx$
  - $\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$
- where C is an arbitrary constant.

SEPARABLE EQUATIONS – Example #1 - SOLN

- We could have used a constant C_1 on the left side and another constant C_2 on
the right side.
  - However, then, we could combine these constants by writing $C = C_2 - C_1$.

SEPARABLE EQUATIONS – Example #1 - SOLN

- Solving for y, we get:

$$y = \sqrt[3]{x^3 + 3C}$$

  - We could leave the solution like this or we could write it in the form $y = \sqrt[3]{x^3 + K}$ where K = 3C.
  - Since C is an arbitrary constant, so is K.

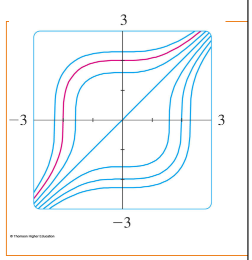
SEPARABLE EQUATIONS – Example #1 - SOLN

- If we put x = 0 in the general solution in (a), we get: $y(0) = \sqrt[3]{K}$
  - To satisfy the initial condition $y(0) = 2$, we must have $\sqrt[3]{K} = 2$, and so K = 8.
  - So, the solution of the initial-value problem is:

$$y = \sqrt[3]{x^3 + 8}$$

SEPARABLE EQUATIONS – Example #1 - SOLN

- The figure shows graphs of several members of the family of solutions of the differential equation in Ex 1.
  - The solution of the initial-value problem in (b) is shown in red.



(B) Example #2

- Given the DE $\frac{dy}{dx} = -\frac{x}{y}$ and $y(2) = 5$
- (a) Solve
- (b) Graph

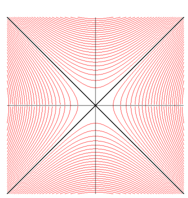
Family of solutions (general solution) of a differential equation

Example

$$\frac{dy}{dx} = \frac{x}{y} \qquad \int y\,dy = \int x\,dx$$

$$y^2 = x^2 + C$$

The picture on the right shows some solutions to the above differential equation. The straight lines $y = x$ and $y = -x$ are special solutions. A unique solution curve goes through any point of the plane different from the origin. The special solutions $y = x$ and $y = -x$ go both through the origin.



SEPARABLE EQUATIONS – Example #3

- Solve the equation
  - $y' = x^2y$
  - First, we rewrite the equation using Leibniz notation: $\frac{dy}{dx} = x^2 y$

SEPARABLE EQUATIONS – Example #3 - SOLN

- If $y \neq 0$, we can rewrite it in differential notation and integrate:

$$\frac{dy}{y} = x^2 dx \qquad y \neq 0$$

$$\int \frac{dy}{y} = \int x^2 dx$$

$$\ln|y| = \frac{x^3}{3} + C$$

SEPARABLE EQUATIONS – Example #3 - SOLN

- The equation defines *y* implicitly as a function of *x*.
- However, in this case, we can solve explicitly for *y*.

$$|y| = e^{\ln|y|} = e^{(x^3/3)+C} = e^C e^{x^3/3}$$

- Hence, $y = \pm e^C e^{x^3/3}$

SEPARABLE EQUATIONS – Example #3 - SOLN

- We can easily verify that the function $y = 0$ is also a solution of the given differential equation.
  - So, we can write the general solution in the form

$$y = Ae^{x^3/3}$$

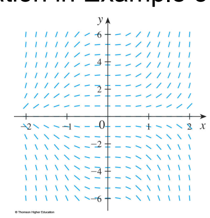
where *A* is an arbitrary constant ($A = e^C$, or $A = -e^C$, or $A = 0$).

SEPARABLE EQUATIONS – Example #3 - SOLN

- The figure shows a direction field for the differential equation in Example 3.

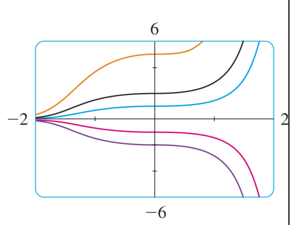
$$y = Ae^{x^3/3}$$

- Compare it with the next figure, in which we use the equation to graph solutions for several values of *A*.



SEPARABLE EQUATIONS – Example #3 - SOLN

- If you use the direction field to sketch solution curves with *y*-intercepts 5, 2, 1, –1, and –2, they will resemble the curves in the figure.



(B) Example #4

- Given the DE $y' = x^2 y$ and $y\left(\sqrt[3]{3}\right) = 4e$
- (a) Solve
- (b) Graph

(B) Example #5

- Given the DE

$$y + y\frac{dy}{dx} = xy - \frac{dy}{dx} \text{ and } y(2) = 1$$

- (a) Solve
- (b) Graph

(B) Example #6

- Given the DE

$$\frac{dy}{dx} = xe^{y-x} \text{ and } y(0) = 2$$

- (a) Solve
- (b) Graph

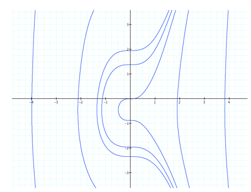
(B) Example #7

- Given the DE $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$
- (a) Solve given $y(1) = \pi$
- (b) Graph the solutions on a slope field diagram

(B) Example #7 – Graphic SOLN

- Here is the graphic solution for

$$\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}$$



Example #8

- Solve $\dfrac{dy}{dx} = 2xy^2$

SOLN to Example #8

A separable differential equation can be expressed as the product of a function of x and a function of y.

$$\frac{dy}{dx} = g(x)\cdot h(y) \qquad h(y) \neq 0$$

Example:

$$\frac{dy}{dx} = 2xy^2$$

Multiply both sides by dx and divide both sides by y^2 to separate the variables. (Assume y^2 is never zero.)

$$\frac{dy}{y^2} = 2x\ dx$$

$$y^{-2}dy = 2x\ dx$$

SOLN to Example #8

A separable differential equation can be expressed as the product of a function of x and a function of y.

$$\frac{dy}{dx} = g(x)\cdot h(y) \qquad h(y) \neq 0$$

Example:

$$\frac{dy}{dx} = 2xy^2$$

$$\frac{dy}{y^2} = 2x\,dx$$

$$y^{-2}dy = 2x\,dx$$

$$\int y^{-2}dy = \int 2x\,dx$$

$$-y^{-1} + C_1 = x^2 + C_2$$

$$-\frac{1}{y} = x^2 + C$$

Combined constants of integration

$$-\frac{1}{x^2+C} = y$$

$$\boxed{y = -\frac{1}{x^2+C}}$$

Example #9

- Solve $\frac{dy}{dx} = 2x\left(1+y^2\right)e^{x^2}$

SOLN to Example #9

$$\frac{dy}{dx} = 2x\left(1+y^2\right)e^{x^2}$$ Separable differential equation

$$\frac{1}{1+y^2}dy = 2x\,e^{x^2}dx$$

$$\int \frac{1}{1+y^2}dy = \int 2x\,e^{x^2}dx \qquad u = x^2,\quad du = 2x\,dx$$

$$\int \frac{1}{1+y^2}dy = \int e^u du$$

$$\tan^{-1}y + C_1 = e^u + C_2$$

$$\tan^{-1}y + C_1 = e^{x^2} + C_2$$

$$\tan^{-1}y = e^{x^2} + C$$ Combined constants of integration

SOLN to Example #9:

$$\frac{dy}{dx} = 2x\left(1+y^2\right)e^{x^2}$$

$$\vdots$$

$$\tan^{-1}y = e^{x^2} + C$$

We now have y as an implicit function of x.

$$\tan\left(\tan^{-1}y\right) = \tan\left(e^{x^2}+C\right)$$

We can find y as an explicit function of x by taking the tangent of both sides.

$$\boxed{y = \tan\left(e^{x^2}+C\right)}$$

(C) - Application - Exponential Growth

- Write a DE for the statement: the rate of growth of a population is directly proportional to the population
- Solve this DE

(C) - Application - Exponential Growth

- Write a DE for the statement: the rate of growth of a population is directly proportional to the population

$$\frac{dP}{dt} \propto P \quad \text{or} \quad \frac{dP}{dt} = kP$$

- Solve this DE:

$$\frac{dP}{dt} = kP$$

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln|P| = kt + C$$

$$P(t) = e^{kt+C} = e^C e^{kt}$$

$$P(t) = Ce^{kt}$$

(D) Examples

- The population of bacteria grown in a culture follows the Law of Natural Growth with a growth rate of 15% per hour. There are 10,000 bacteria after the first hour.

- (a) Write an equation for P(t)
- (b) How many bacteria will there be in 4 hours?
- (c) when will the number of bacteria be 250,000?

(D) Examples

- The concentration of phosphate pollutants in a lake follows the Law of Natural Growth with a decay rate of 5.75% per year. The phosphate pollutant concentrations are 125 ppm in the second year.

- (a) Write an equation for P(t)
- (b) What will there be phosphate pollutant concentration in 10 years?
- (c) A given species of fish can be re-introduced into the lake when the phosphate concentration falls below 35 ppm. When can the fish be re-introduced?

Challenge Problems

- Solve these DEs:

(i) $\frac{dy}{dx} = \frac{xy + 3y + 2x + 6}{xy - 2y - x + 2}$

(ii) $e^y \sin 2x dx + \cos\left(e^{2y} - y\right) dy = 0$

(iii) $x \sin x dx + \left(1 + 4y^3\right) dy = 0$

Example:9. Solve the differential equation

$$\frac{dy}{dx} = \frac{xy + 3y + 2x + 6}{xy - 2y - x + 2}$$

Solution:

$$\frac{dy}{dx} = \frac{xy + 3y + 2x + 6}{xy - 2y - x + 2}$$

$$= \frac{y(x+3) + 2(x+3)}{y(x-2) - 1(x-2)}$$

$$= \frac{(x+3)(y+2)}{(x-2)(y-1)}$$

$$\left[1 - \frac{3}{y+2}\right] dy = \left[1 + \frac{5}{x-2}\right] dx$$

$$\int \left(1 - \frac{3}{y+2}\right) dy = \int \left(1 + \frac{5}{x-2}\right) dx$$

$$y - 3\ln|y+2| = x + 5\ln|x-2| + c\,.$$

$$y - 3\ln|y+2| = x + 5\ln|x-2| + c\,.$$

Example:10. Solve the differential equation

$$e^y \sin 2x dx + \cos x\left(e^{2y} - y\right) dy = 0$$

Solution:

$$e^y \sin 2x dx + \cos x\left(e^{2y} - y\right) dy = \circ$$

separating the Variables by dividing differential equation by $e^y \cos x$

$$\frac{\sin 2x}{\cos x} dx + \frac{e^{2y} - y}{e^y} dy = 0$$

$$\frac{2\sin x \cos x}{\cos x} dx + e^{-y}\left(e^{2y} - y\right) dy = 0$$

$$2\sin x dx + \left(e^y - ye^{-y}\right) dy = 0$$

$$\int 2\sin x dx + \int \left(e^y - ye^{-y}\right) dy = 0$$

$$-2\cos x + e^y - \left(-ye^{-y} - e^{-y}\right) = c$$

$$-2\cos x + e^y + ye^{-y} + e^{-y} = c$$

Example:11. Solve the IVP

$$x \sin x dx + \left(1 + 4y^3\right) dy = 0 \quad ; \quad y(\pi) = 0\,.$$

Solution:

$$x \sin x dx = -\left(1 + 4y^3\right) dy$$

$$\int x \sin x dx = -\int \left(1 + 4y^3\right) dy$$

$$\sin x - x\cos x = -y - y^4 + c$$

$$x = \pi, y = 0$$

$$\sin\pi - \pi\cos\pi = c \qquad \sin\pi = 0, \cos\pi = -1$$

$$c = \pi$$

Solution of IVP is $\sin x - x\cos x + y + y^4 = \pi.$

MIXING PROBLEMS

- A tank contains 20 kg of salt dissolved in 5000 L of water.
  - Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min.
  - The solution is kept thoroughly mixed and drains from the tank at the same rate.
  - How much salt remains in the tank after half an hour?

MIXING PROBLEMS

- Let $y(t)$ be the amount of salt (in kilograms) after t minutes.
- We are given that $y(0) = 20$ and we want to find $y(30)$.
  - We do this by finding a differential equation satisfied by $y(t)$.

MIXING PROBLEMS

- Note that dy/dt is the rate of change of the amount of salt.
- Thus, $\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$

  where:
  - 'Rate in' is the rate at which salt enters the tank.
  - 'Rate out' is the rate at which it leaves the tank.

RATE IN

- We have:

$$\text{rate in} = \left(0.03\frac{\text{kg}}{\text{L}}\right)\left(25\frac{\text{L}}{\text{min}}\right) = 0.75\frac{\text{kg}}{\text{min}}$$

MIXING PROBLEMS

- The tank always contains 5000 L of liquid.
  - So, the concentration at time t is $y(t)/5000$ (measured in kg/L).

RATE OUT

- As the brine flows out at a rate of 25 L/min, we have:

$$\text{rate out} = \left(\frac{y(t)}{5000}\frac{\text{kg}}{\text{L}}\right)\left(25\frac{\text{L}}{\text{min}}\right) = \frac{y(t)}{200}\frac{\text{kg}}{\text{min}}$$

MIXING PROBLEMS

- Thus, from Equation 5, we get:

$$\frac{dy}{dt} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

  - Solving this separable differential equation, we obtain: $\int \frac{dy}{150 - y} = \int \frac{dt}{200}$

$$-\ln|150 - y| = \frac{t}{200} + C$$

MIXING PROBLEMS

- Since $y(0) = 20$, we have:
  $-\ln 130 = C$

So, $-\ln|150 - y| = \dfrac{t}{200} - \ln 130$

MIXING PROBLEMS

- Therefore, $|150 - y| = 130e^{-t/200}$

  - $y(t)$ is continuous and $y(0) = 20$, and the right side is never 0.
  - We deduce that $150 - y(t)$ is always positive.

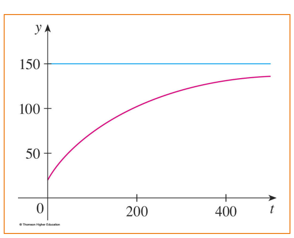
MIXING PROBLEMS

- Thus, $|150 - y| = 150 - y$.
- So, $y(t) = 150 - 130e^{-t/200}$
  - The amount of salt after 30 min is:

$$y(30) = 150 - 130e^{-30/200} \approx 38.1 \text{ kg}$$

MIXING PROBLEMS

- Here's the graph of the function $y(t)$ of Example 6.
  - Notice that, as time goes by, the amount of salt approaches 150 kg.



Logistic Growth Model

Real-life populations do not increase forever. There is some limiting factor such as food or living space.

There is a maximum population, or carrying capacity, M.

A more realistic model is the logistic growth model where growth rate is proportional to both the size of the population (y) and the amount by which y falls short of the maximal size (M-y). Then we have the equation:

$$\frac{dy}{dt} = ky(M - y)$$

The solution to this differential equation:

$$y = \frac{y_0 M}{y_0 + (M - y_0)e^{-kMt}}, \quad \text{where } y_0 = y(0)$$