Lesson 52 – Integration by Substitution

IBHL - Calculus - Santowski

Lesson Objectives

- Use the method of substitution to integrate simple composite power, exponential, logarithmic and trigonometric functions both in a mathematical context and in a real world problem context

Fast Five

- Differentiate the following functions:

$$\frac{d}{dx}\left(x^2+5\right)^3$$

$$\frac{d}{dx}e^{x^2}$$

$$\frac{d}{dx}\sin\left(\frac{4}{x^3}\right)$$

$$\frac{d}{dx}\ln\left(x^3+1\right)$$

$$\frac{d}{dx}\sin\left(\ln\left(\sqrt{x+3}\right)\right)$$

$$\frac{d}{dx}\frac{1}{\left(x^2+6x\right)^2}$$

(A) Introduction

- At this point, we know how to do simple integrals wherein we simply apply our standard integral "formulas"
- But, similar to our investigation into differential calculus, functions become more difficult/challenging, so we developed new "rules" for working with more complex functions
- Likewise, we will see the same idea in integral calculus and we shall introduce 2 methods that will help us to work with integrals

(B) "Simple" Examples ????

- Find the following:

$$\int \sqrt[4]{x}\,dx$$

$$\int \frac{1}{t^3}\,dt$$

$$\int \cos w\,dw$$

$$\int e^y\,dy$$

- Now, try these:

$$\int e^{x^2}dx$$

$$\int \sin\left(\frac{4}{x^3}\right)dx$$

(C) Looking for Patterns

- Alright, let's use wolframalpha to help us with some of the following integrals:
- Now, look at our fast 5
- Now, let's look for patterns???
- Examples

$$\int 2xe^{x^2}dx$$

$$\int -\frac{12}{x^4}\sin\left(\frac{4}{x^3}\right)dx$$

$$\int \frac{3x^2}{x^3+1}dx$$

$$\int \frac{\cos\left(\ln(x)\right)}{x}dx$$

(C) Looking for Patterns

- So, in all the integrals presented here, we see that some part of the function to be integrated is a COMPOSED function and then the second pattern we observe is that we also see some of the derivative of the "inner" function appearing in the integral
- Here are more examples to illustrate our "pattern"

$$\int 9x^2\sqrt[4]{6x^3+5}\,dx$$

$$\int\left(1-\frac{1}{w}\right)\cos(w-\ln w)dw$$

$$\int(16y-2)e^{4y^2-y}dy$$

(D) Generalization from our Pattern

- So we can make the following generalization from our observation of patterns:

$$\int f(g(x))\cdot g'(x)\,dx=\int f(u)\,du$$

where $u=g(x)$ and then $du=g'(x)dx$

- But the question becomes: how do we know what substitution to make???
- Generalization: ask yourself what portion of the integrand has an inside function and can you do the integral with that inside function present. If you can't then there is a pretty good chance that the inside function will be the substitution.

(E) Working out Some Examples

In these problems, a substitution is given.

1. $\int(3x-5)^{17}\,dx,\ u=3x-5$
2. $\int_0^4 x\sqrt{x^2+9}\,dx,\ u=x^2+9$
3. $\int\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx,\ u=\sqrt{x}.$
4. $\int\frac{\cos 3x\,dx}{5+2\sin 3x},\ u=5+2\sin 3x$

(E) Working out Some Examples

- Integrate $\int x^2\left(3-10x^3\right)^4 dx$
- Integrate $\int 9x^2\sqrt[4]{6x^3+5}\,dx$

(E) Working out Some Examples

- Integrate $\int x^2\left(3-10x^3\right)^4 dx$

Let $u=3-10x^3$ then $du=-30x^2dx$

so $x^2dx=-\frac{1}{30}du$ and we get :

$$=\int u^4\cdot-\frac{1}{30}du=-\frac{1}{30}\int u^4du$$

$$=-\frac{1}{30}\cdot\frac{u^{4+1}}{4+1}+C$$

$$=-\frac{1}{150}\left(3-10x^3\right)^5+C$$

(E) Working out Some Examples

- Integrate $\int 9x^2\sqrt[4]{6x^3+5}\,dx$

Let $u=6x^3+5$ then $du=18x^2dx$

so $9x^2dx=\frac{1}{2}du$ and we get :

$$=\int\sqrt[4]{u}\cdot\frac{1}{2}du=\frac{1}{2}\int u^{1/4}du$$

$$=\frac{1}{2}\cdot\frac{u^{1/4+1}}{1/4+1}+C$$

$$=\frac{2}{5}\left(6x^3+5\right)^{\frac{5}{4}}+C$$

(E) Working out Some Examples

In these problems, you need to determine the substitution yourself.

5. $\int (4-3x)^7\,dx.$

6. $\int_{\pi/4}^{\pi/3} \csc^2(5x)\,dx$

7. $\int x^2 e^{3x^3-1}\,dx$

(F) Further Examples

- Integrate the following:

$\int x^2 e^{x^3}\,dx$ $\qquad$ $\int \frac{x^2}{\sqrt{1-x^3}}dx$

$\int \frac{\ln x}{x}dx$ $\qquad$ $\int \sin^4(x)\cos(x)dx$

$\int \sin x \cos x dx$ $\qquad$ $\int \tan x dx$

$\int \cos(3x)\sin^{10}(3x)dx$ $\qquad$ $\int x^2 \sin(x^3)dx$

(F) Further Examples

- Integrate the following:

$$\int_0^{\pi/4} \tan x \sec^2 x\,dx$$

$$\int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$$

$$\int_{-1}^1 x^2\sqrt{x^3+1}\,dx$$

(F) Challenge Examples

- Integrate the following:

(a) $\int \frac{x}{x+1}\,dx$

(b) $\int \sin^2(x)\,dx$

(c) $\int \cos^2(x)\,dx$

(d) $\int \sec(x)\,dx$

(F) Challenge Examples

Sometimes there is more than one way to skin a cat:

8. Find $\int \frac{x}{1+x}\,dx$, both by long division and by substituting $u = 1+x$.

9. Find $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$, both by substituting $u = z^2+1$ and $u = \sqrt[3]{z^2+1}$.

(F) Challenge Examples

- Integrate the following:

(a) $\int \frac{x^2}{x+1}\,dx$

(b) $\int \frac{x^4+x-4}{x^2-2}\,dx$ $\qquad$ (c) $\int \frac{x^5-35x}{x^2+6}\,dx$

(d) $\int \frac{dx}{x^2-4x+4}$ $\qquad$ (e) $\int \frac{dx}{\sqrt{-x^2+4x-3}}$

(f) $\int \frac{dx}{2+9x^2}$ $\qquad$ (g) $\int \frac{dx}{\sqrt{4-25x^2}}$

An Application to Business

In 1990 the head of the research and development department of the Soloron Corp. claimed that the cost of producing solar cell panels would drop at the rate of

$$\frac{58}{(3t+2)^2}, 0 \le t \le 10$$

dollars per peak watt for the next t years, with t = 0 corresponding to the beginning of the year 1990. (A peak watt is the power produced at noon on a sunny day.) In 1990 the panels, which are used for photovoltaic power systems, cost $10 per peak watt. Find an expression giving the cost per peak watt of producing solar cell panels at the beginning of year t. What was the cost at the beginning of 2000?

An Application to Business

This tells you the expression is a derivative.

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Since the expression is a dropping rate in cost, the expression is C'(x) or the derivative of the cost C(x) and it should be negative since it is dropping. Thus:

$$C'(t) = \frac{-58}{(3t+2)^2}, 0 \le t \le 10$$

$$\int \frac{-58}{(3t+2)^2}dt = \int \frac{-58}{u^2}\frac{du}{3} = \frac{-58}{3}\int u^{-2}du = \frac{-58}{3}\frac{u^{-1}}{-1} = \frac{58}{3u} = \frac{58}{3(3t+2)} + C$$

$$u = 3t + 2$$

$$du = 3dt$$

$$\frac{du}{3} = dt$$

The Cost Function C(t) is $C(t) = \frac{58}{3(3t+2)} + C$

The Cost Function C(t) is $C(t) = \frac{58}{3(3t+2)} + C$

Use the initial condition that the cost in 1990 was $10, or when t = 0, C(0)= $10, thus

$$10 = \frac{58}{3(3\cdot 0+2)} + C$$

$$10 = \frac{58}{6} + C$$

$$60 = 58 + 6C$$

$$2 = 6C$$

$$\frac{2}{6} = \frac{1}{3} = C$$

Hence, $C(t) = \frac{58}{3(3t+2)} + \frac{1}{3}$

$$C(t) = \frac{58}{3(3t+2)} + \frac{1}{3}$$

Now to find the cost per peak watt at the beginning of the year 2000 which is 10 years from 1990 and would correspond to t = 10.

$$C(10) = \frac{58}{3(3(10)+2)} + \frac{1}{3} = \frac{58}{3(32)} + \frac{1}{3}$$

$$\frac{58}{96} + \frac{1}{3} \approx .9375 \approx .94$$

Thus the cost per peak watt of producing solar cell panels at the beginning of 2000 is approximately $.94 per peak watt.

Applications

The marginal price of a supply level of x bottles of baby shampoo per week is given by

$$\mathbf{p'(x) = \frac{300}{(3x + 25)^2}}$$

Find the price-supply equation if the distributor of the shampoo is willing to supply 75 bottles a week at a price of $1.60 per bottle.

To find p (x) we need the ∫ p ‘ (x) dx

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Applications - continued

The marginal price of a supply level of x bottles of baby shampoo per week is given by $p'(x) = \frac{300}{(3x+25)^2}$

Find the price-supply equation if the distributor of the shampoo is willing to supply 75 bottles a week at a price of $1.60 per bottle.

$$\int \frac{300}{(3x+25)^2}\,dx = 300\int (3x+25)^{-2}\,dx$$

Let u = 3x + 25 and du = 3 dx

$$300\int (3x+25)^{-2}\,dx = \frac{300}{3}\int (3x+25)^{-2}\,3dx = 100\int u^{-2}\,du =$$

$$-100u^{-1} + c$$

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Applications - continued

The marginal price of a supply level of x bottles of baby shampoo per week is given by $p'(x) = \frac{300}{(3x+25)^2}$

Find the price-supply equation if the distributor of the shampoo is willing to supply 75 bottles a week at a price of $1.60 per bottle.

With u = 3x + 25, $-100u^{-1} + c = -100(3x+25)^{-1} + c$

so $p(x) = -100(3x+25)^{-1} + c = \frac{-100}{3x+25} + c$

Remember you may differentiate to check your work!

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Applications - continued

The marginal price of a supply level of x bottles of baby shampoo per week is given by $p'(x) = \frac{300}{(3x+25)^2}$

Find the price-supply equation if the distributor of the shampoo is willing to supply 75 bottles a week at a price of $1.60 per bottle.

$p(x) = \frac{-100}{3x+25} + c$ Now we need to find c using the fact

That 75 bottles sell for $1.60 per bottle.

$1.60 = \frac{-100}{3(75)+25} + c$ and c = 2

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Applications - concluded

The marginal price of a supply level of x bottles of baby shampoo per week is given by $p'(x) = \frac{300}{(3x+25)^2}$

Find the price-supply equation if the distributor of the shampoo is willing to supply 75 bottles a week at a price of $1.60 per bottle.

So $p(x) = \frac{-100}{3x+25} + 2$

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Further Substitutions

- Given the ellipse $4x^2 + y^2 = 4$, determine:
- (a) the x-intercepts
- (b) the area between the ellipse, the x-axis and the zeroes

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Further Substitutions

- Integrate the following indefinite integrals:

(a) $\int \sqrt{16-x^2}\,dx$

(b) $\int \sqrt{x^2-16}\,dx$

(c) $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

(d) $\int \frac{dx}{x^2\sqrt{x^2+4}}$

(e) $\int \frac{x}{\sqrt{x^2+4}}\,dx$

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CHALLENGE

- Evaluate:

$$\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{\left(4x^2+9\right)^{3/2}}\,dx$$

- ANS = 3/32