Lesson 51 (DAY 2) – Volumes of Revolution – Method of Cylinders

Calculus – Santowski

Investigation

- Draw the curve of $f(x) = x - x^2$ between x = 0 and x = 1
- Draw a rectangle whose height is $f(x_i)$ and whose width is Δx
- Visualize what happens to this rectangle if it is rotated around the x-axis. Tell me what "shape" results
- Tell me what the idea of "rectangle is perpendicular to the rotation axis" means
- Now visualize what happens to this rectangle if it is rotated around the y-axis. Tell me what "shape" results
- Tell me what the idea of "rectangle is parallel to the rotation axis" means

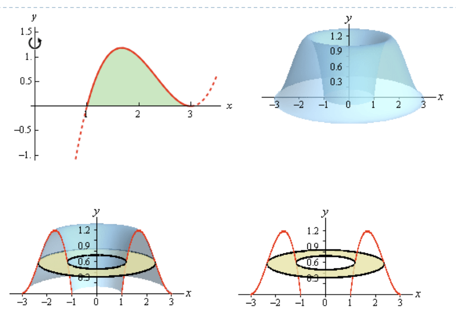
(A) Example #1

- We will work with the example of $y = x - x^2$ between x = 0 and x = 1 and rotating around the y-axis
- This example is explained and visualized in the following link from Visual Calculus:
- http://archives.math.utk.edu/visual.calculus/5/volumes.6/index.html

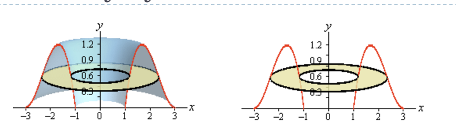
Fast Five

- Graph $y = (x-1)(x-3)^2$ on the interval x = 0 to x = 4
- Shade in the region bounded the x-axis and the curve and determine the volume of revolution formed when the region is rotated around the y-axis

Fast Five



Volume by Cylinders

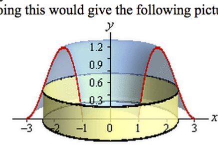


This leads to several problems. First, both the inner and outer radius are defined by the same function. This, in itself, can be dealt with on occasion as we saw in a example in the **Area Between Curves** section. However, this usually means more work than other methods so it's often not the best approach.

This leads to the second problem we got here. In order to use rings we would need to put this function into the form $x = f(y)$. That is NOT easy in general for a cubic polynomial and in other cases may not even be possible to do. Even when it is possible to do this the resulting equation is often significantly messier than the original which can also cause problems.

Volume by Cylinders

What we need to do is to find a different way to cut the solid that will give us a cross-sectional area that we can work with. One way to do this is to think of our solid as a lump of cookie dough and instead of cutting it perpendicular to the axis of rotation we could instead center a cylindrical cookie cutter on the axis of rotation and push this down into the solid. Doing this would give the following picture,



(B) General Formula

- Let $f(x)$ be a function which is continuous on the closed interval [a, b]. The volume of the solid obtained by rotating the graph of $f(x)$ from x = a to x = b about the y-axis is the integral

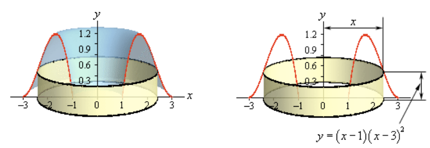
$$V = 2\pi \int_a^b x\, f(x)\, dx$$

Examples

- The following in class examples are from the following web site, which you should visit if you do not understand our in-class discussion:
- http://tutorial.math.lamar.edu/Classes/CalcI/VolumeWithCylinder.aspx

(C) Example 1

- Determine the volume of the solid produced when the region between $y = (x-1)(x-3)^2$ and the x-axis is rotated about the y-axis



(D) Example #2

- Determine the volume of the solid produced when the region between y = x – 1, and $y = 2\sqrt{x-1}$ is rotated about the line x = 6

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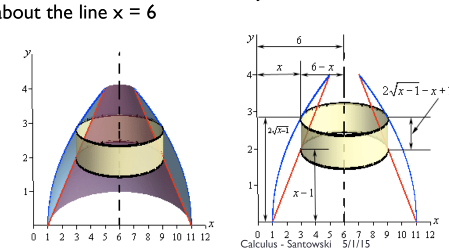

(E) Example #3

- Determine the volume of the solid produced when the region between $y = \sqrt[3]{x}$, x=8, and the x-axis is rotated about the x-axis

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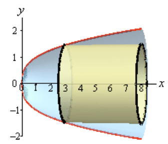
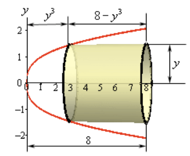




(F) Homework

- Link to WorksheetQuestions 39,40,41,42,43a,44b,47,46

Further Examples – Day 2

- Ex 1.
- The base of a solid is a region between the parabolas $x = y^2$ and $x = 3 - 2y^2$. Find the volume of the solid given that the cross section perpendicular to the x-axis are:
- (a) rectangles of height *h*
- (b) equilateral triangles
- (c) isosceles right triangles, hypotenuse on the *xy* plane

Further Examples – Day 2

- Ex 2.
- Find the volume enclosed by the surface obtained by revolving the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ about the x-axis.
- (a) Use the method of discs
- (b) Use the method of cylinders

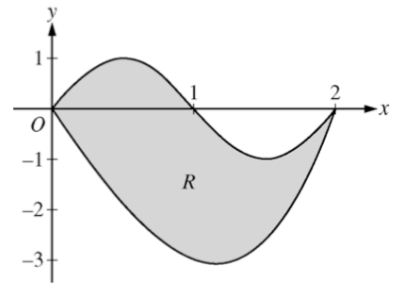
Further Examples – Day 2

- Ex 3.
- Let $f(x) = \begin{cases} \sqrt{3x}, & 0 \le x < 1 \\ \sqrt{4 - x^2}, & 1 \le x \le 2 \end{cases}$
- And let Ω be the region between the graph of *y* = *f*(*x*) and the *x*-axis
- (a) Revolve Ω about the *y*-axis and express the volume of the resulting solid as an integral in *x*
- (b) Revolve Ω about the *y*-axis and express the volume of the resulting solid as an integral in *y*
- (c) Calculate the volume by evaluating one of these integrals

Further Examples – Day 2

- Ex 4.
- The region Ω in the first quadrant bounded by the parabola $y = r^2 - x^2$ and the co-ordinate axis is revolved around the y-axis. The resulting solid is called a paraboloid. A vertical hole of radius *a*, where *a* < *r*, centered along the *y*-axis is drilled through the paraboloid. Find the volume of the solid that remains by:
- (a) integrating wrt *x*
- (b) integrating wrt *y*

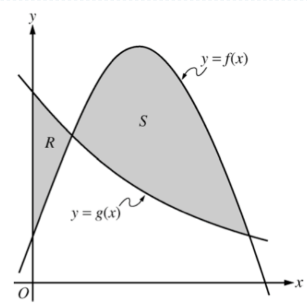
AP Calc AB Exam 2008, Section 2, Part A, Q1



AP Calc AB Exam 2008, Section 2, Part A, Q1

- Let R be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$ as shown in the figure above.
- (a) Find the area of R.
- (b) The horizontal line y = 2 splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.
- (c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.
- (d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$ Find the volume of water in the pond.

AP Calc AB Exam 2005, Section 2, Part A, Q1



AP Calc AB Exam 2005, Section 2, Part A, Q1

Let f and g be the functions given by $f(x) = \frac{1}{4} + \sin(\pi x)$ and $g(x) = 4^{-x}$. Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of f and g, and let S be the shaded region in the first quadrant enclosed by the graphs of f and g, as shown in the figure above.

(a) Find the area of R.

(b) Find the area of S.

(c) Find the volume of the solid generated when S is revolved about the horizontal line $y = -1$.