Lesson 49 – Working with Definite Integrals

HL Math- Santowski

Lesson Objectives

- 1. Calculate simple definite integrals
- 2. Calculate definite integrals using the properties of definite integrals
- 3. Determine total areas under curves
- 4. Apply definite integrals to a real world problems

(A) Review

- But we have introduced a new symbol for our antiderivative → the integral symbol (∫)
- So now we can put our relationships together

$$A(x) = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

- Where $\int_a^b f(x)dx$ is now referred to as a definite integral

(B) The Fundamental Theorem of Calculus

- The fundamental theorem of calculus shows the connection between antiderivatives and definite integrals
- Let f be a continuous function on the interval $[a,b]$ and let F be any antiderivative of f. Then $\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$

(C) Examples

- Let's work with the FTC on the following examples:

$$(i)\ \int_2^5 \left(5x^2 - 4x + 5\right)dx$$

$$(ii)\ \int_1^2 \frac{dy}{y}$$

$$(iii)\ \int_0^{\pi} \sin x dx$$

$$(iv)\ \int_0^{\frac{\pi}{2}} \left(2e^{2x} - 5\cos x\right)dx$$

(C) Examples

- Evaluate the following integrals:

$$(a)\ \int_0^{\frac{\pi}{3}} \left(\frac{2}{\pi}x - 2\sec^2 x\right)dx$$

$$(b)\ \int_0^{\frac{\pi}{2}} (\cos\theta + 2\sin\theta)d\theta$$

$$(c)\ \int_{-e}^{-e^2} \frac{3}{x}dx$$

$$(d)\ \int_{\ln 6}^{\ln 3} 8e^t dt$$

(D) Properties of Definite Integrals

- To further develop definite integrals and there applications, we need to establish some basic properties of definite integrals:

$$(i)\ \int_a^a f(x)dx = 0$$

$$(ii)\ \int_a^b (k \times f(x))dx = k \times \int_a^b f(x)dx$$

$$(iii)\ \int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$(iv)\ \int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$(v)\ \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

(E) Further Examples

- Let's work with the FTC on the following examples.

$$(i)\ \int_0^4 \left(x^2 - 4\right)dx$$

$$(ii)\ \int_0^4 \left(x^2 - 3x\right)dx$$

$$(iii)\ \int_0^4 \left|x^2 - 4\right|dx$$

(*iv*) Find the TOTAL between the x-axis and the function y = x^2 - 3x

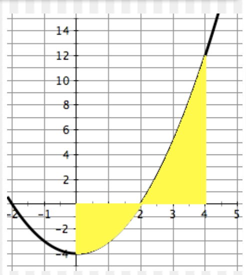
(E) Further Examples

- Let's work with the FTC on the following examples

$$(i)\ \int_0^2 \left(x^2 - 4\right)dx$$

$$(ii)\ \int_1^3 \left(x^2 - 3x\right)dx$$

$$(iii)\ \int_0^4 \left(x^2 - 4\right)dx$$



(E) Further Examples

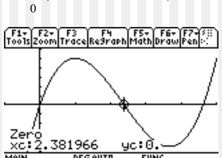
Evaluate and interpret the definite integral in terms of areas.

$$\int_0^4 \left(x^3 - 7x^2 + 11x\right)dx$$

(E) Further Examples

Interpret the definite integral in terms of areas.

$$\int_0^4 \left(x^3 - 7x^2 + 11x\right)dx$$



(E) Further Examples

- So the answer to the question of interpreting the definite integral $\int_0^4 \left(x^3 - 7x^2 + 11x\right)dx$ in terms of areas is:
- (i) 2.67 if the interpretation is as the difference between the positive area and the negative area (NET area)
- (ii) 12.77 if we interpret the answer to mean the TOTAL area of the regions

(E) Further Examples

- We are going to set up a convention here in that we would like the area to be interpreted as a positive number when we are working on TOTAL Area interpretations of the DI
- So, we will take the absolute value of negative "areas" or values
- Then the second consideration will be when a function has an x-intercept in the interval $[a,b]$ → we will then break the area into 2 (or more) sub-intervals $[a,c]$ & $[c,b]$ and work with 2 separate definite integrals

(F) DEFINITE INTEGRALS & DISCONTINUOUS FUNCTIONS

- Evaluate the following definite integrals:

$$(a)\ \int_2^5 |x-3|\,dx$$

$$(b)\ \int_{-2}^{2} |x^2-1|\,dx$$

$$(c)\ \int_{-2}^{4} f(x)\,dx \text{ where } f(x)=\begin{cases} 2+x^2 & x<0 \\ \frac{1}{2}x+2 & x\geq 0 \end{cases}$$

$$(d)\ \int_0^{\frac{3\pi}{2}} f(x)\,dx \text{ where } f(x)=\begin{cases} 2\sin x & x\leq \frac{\pi}{2} \\ 2+\cos x & x>\frac{\pi}{2} \end{cases}$$

(F) DEFINITE INTEGRALS & DISCONTINUOUS FUNCTIONS

- True or false & explain your answer:

$$\int_0^{2\pi} \sec^2 x\,dx = \tan x\Big|_0^{2\pi} = 0-0=0$$

(G) APPLICATIONS OF DEFINITE INTEGRALS – MOTION PROBLEMS

- 1. An object starts at the origin and moves along the x-axis with a velocity $v(t) = 10t - t^2$ for $0 \leq t \leq 10$
- (a) What is the position of the object at any time, t ?
- (b) What is the position of the object at the moment when it reaches its maximum velocity?
- (c) How far has the object traveled before it starts to decelerate?

(G) APPLICATIONS OF DEFINITE INTEGRALS – MOTION PROBLEMS

- 1. For the velocity functions given below for a particle moving along a line, determine the distance traveled and displacement of the particle:
- (a) $v(t) = 3t - 5,\ 0 \leq t \leq 3$
- (b) $v(t) = t^2 - 2t - 8,\ 1 \leq t \leq 6$
- 2. The acceleration function and the initial velocity are given for a particle moving along a line. Determine (a) the velocity at time t and (b) the distance traveled during the given time interval:
- (a) $a(t) = t + 4,\ v(0) = 5,\ 0 \leq t \leq 10$
- (b) $a(t) = 2t + 3,\ v(0) = -4,\ 0 \leq t \leq 3$

(G) APPLICATIONS OF DEFINITE INTEGRALS – MOTION PROBLEMS

- Two cars, who are beside each other, leave an intersection at the same instant. They travel along the same road. Car A travels with a velocity of $v(t) = t^2 - t - 6$ m/s while Car B travels with a velocity of $v(t) = 0.5t + 2$ m/s.
- (a) What are the initial velocities of the cars?
- (b) How far has each car gone after 4 seconds have elapsed?
- (c) When are the two cars beside each other again (i.e. when does the trailing car catch up to the leading car?)

(H) AREA BETWEEN CURVES

- Find the area of the regions bounded:
- (a) below f(x) = x + 2 and above $g(x) = x^2$
- (b) by f(x) = 4x and $g(x) = x^3$ from x = -2 to x = 2
- (c) Evaluate $\int_{-1}^{3}\left(x^2-2x\right)dx$ and interpret the result in terms of areas. Then find the area between the graph of $f(x) = x^2 - 2x$ and the x-axis from x = -1 to x = 3

(H) AREA BETWEEN CURVES

- Sketch the curve of $f(x) = x^3 - x^4$ between x = 0 and x = 1.
- (a) Draw a vertical line at $x = k$ such that the region between the curve and axis is divided into 2 regions of equal area. Determine the value of k.
- (b) Draw a horizontal line at $y = h$ such that the region between the curve and axis is divided into 2 regions of equal area. Estimate the value of h. Justify your estimation.