Lesson 45 - Antiderivatives and the Rules of Integration

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HL Math 1 - Calculus - Santowski

A function F is an antiderivative of f on an interval I if $F'(x)=f(x)$ for all x in I.

Let's use an example to figure out what this statement means…

Suppose we know $f(x) = 2x$ and we want to find its antiderivative, F.

If $f(x) = 2x$, then $F'(x) = 2x$. So we know the derivative of F.

Think backwards, what function has a derivative equal to 2x ?

$F(x) = x^2$!!!

To find the antiderivative, do the reverse of finding the derivative.

Is $F(x) = x^2$ the only function whose derivative is 2x ? Or in other words, is $F(x) = x^2$ the only antiderivative of 2x ?

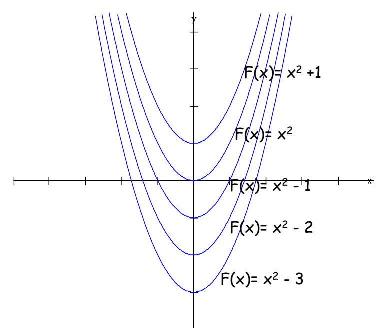
Theorem 1: Let G be an antiderivative of a function f. Then, every antiderivative F of f must be of the form $F(x) = G(x) + C$, where C is a constant.

So the answer to the question is $F(x) = x^2$ the only antiderivative of 2x is NO!!

Example 1: Let $F(x) = x^2 + 4$ and let $G(x) = x^2 - 1$
Then $F'(x) = 2x$ and $G'(x)= 2x$

Thus both F and G are antiderivatives of $f(x) = 2x$.
Note two functions which have the same derivative will only differ by the constant.

This means the antiderivative of a function is a family of functions as pictured on the next slide.



When you integrate (which is another name for finding the antiderivative) you get a family of curves. Each of these is an antiderivative of $f(x)=2x$.

The Indefinite Integral

The process of finding all antiderivatives of a function is called antidifferentiation, or integration.

$\int$ is the symbol used for integration and is called the integral symbol.

We write $\int f(x)dx = F(x) + C$

dx → is the differential of x which denotes the variable of integration

This is called an *indefinite integral*, $f(x)$ is called the integrand and C is called the constant of integration.

<u>Basic Integration Rules</u>

Rule 1: $\int k\,dx = kx + C$ (k, a constant)

Keep in mind that integration is the reverse of differentiation. What function has a derivative k?

kx + C, where C is any constant.

Another way to check the rule is to differentiate the result and see if it matches the integrand. Let's practice.

Example 2:

$\int 2dx = 2x + C$

Example 3:

$\int \pi^2 dx = \pi^2 x + C$

Before we list Rule 2, let's go back and think about derivatives.

When we used the power rule to take the derivative of a power, we multiplied by the power and subtracted one from the exponent.

Example:

$$\frac{d}{dx}\left(x^3\right)=3x^2$$

Since the opposite of multiplying is dividing and the opposite of subtracting is adding, to integrate we'd do the opposite. So, let's try adding 1 to the exponent and dividing by the new exponent.

Integrating: $\int x^3dx=\frac{x^{3+1}}{3+1}=\frac{1}{4}x^4$

Check by differentiating the result: $\frac{d}{dx}\left(\frac{1}{4}x^4\right)=\frac{4}{4}x^3=x^3$

Since we get the integrand we know it works.

Basic Integration Rules

Rule 2: The Power Rule $\int x^n=\frac{x^{n+1}}{n+1}+C \quad n\neq -1$

Example 4: Find the indefinite integral $\int t^3dt$

Solution: $\int t^3dt=\frac{t^4}{4}+C$

Example 5: Find the indefinite integral $\int x^{\frac{3}{2}}dx$

Solution: $\int x^{\frac{3}{2}}dx=\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}+C=\frac{X^{\frac{5}{2}}}{\frac{5}{2}}+C=\frac{2}{5}X^{\frac{5}{2}}+C$

Here are more examples of Rule 1 and Rule 2.

Example 6: Find the indefinite integral $\int\frac{1}{x^3}dx$

Solution: $\int\frac{1}{x^3}dx=\int x^{-3}=\frac{x^{-3+1}}{-3+1}+C=\frac{x^{-2}}{-2}+C=\frac{-1}{2x^2}+C$

Example 7: Find the indefinite integral $\int 1dx$

Solution: $\int 1dx=x+C$

Example 8: Find the indefinite integral $\int -3x^{-2}dx$

Solution: $\int -3x^{-2}dx=-3\int x^{-2}=\frac{-3x^{-2+1}}{-2+1}+C=\frac{-3x^{-1}}{-1}+C=\frac{3}{x}+C$

Basic Integration Rules

Rule 3: The Indefinite Integral of a Constant Multiple of a Function

$$\int cf(x)dx=c\int f(x)dx,\ \text{c is a constant}$$

Rule 4: The Sum Rule (or difference)

$$\int\left[f(x)+g(x)\right]dx=\int f(x)dx+\int g(x)dx$$
$$\int\left[f(x)-g(x)\right]dx=\int f(x)dx-\int g(x)dx$$

Rule 5: $\int e^xdx=e^x+C$

Rule 6: $\int\frac{1}{x}dx=\ln|x|+C$

To check these 2 rules, differentiate the result and you'll see that it matches the integrand.

Integrate and check your answer by taking the derivative.

(a) $\int 3+x^4+e^x\,dx$

(b) $\int(2x-\frac{1}{x}+\frac{3}{x^2}+\sqrt{x}+3e^x)dx$

(c) $\int(2x-4)\left(\frac{1}{x^2}+\frac{3}{x}\right)dx$

Here is the solution for Q(a)in detail.

(a) $\int 3+x^4+e^x\,dx=\int 3dx+\int x^4dx+\int e^xdx$

$$=3x+\frac{x^{4+1}}{4+1}+e^x+C$$

$$=3x+\frac{1}{5}x^5+e^x+C$$

Q(b) Integrate $\int (2x - \frac{1}{x} + \frac{3}{x^2} + \sqrt{x} + 3e^x)dx$

Using the sum rule we separate this into 5 problems.

$$\int 2xdx - \int \frac{1}{x}dx + \int \frac{3}{x^2}dx + \int \sqrt{x}dx + \int 3e^x dx$$

ANSWER:

$$\int 2xdx - \int \frac{1}{x}dx + \int \frac{3}{x^2}dx + \int \sqrt{x}dx + \int 3e^x dx = x^2 - \ln|x| - \frac{3}{x} + \frac{2}{3}x^{\frac{3}{2}} + 3e^x + C$$

You may be wondering why we didn't use the C before now. Let's say that we had five constants $C_1 + C_2 + C_3 + C_4 + C_5$. Now we add all of them together and call them C. In essence that's what's going on above.

(c) Integrate $\int (2x-4)\left(\frac{1}{x^2} + \frac{3}{x}\right)dx$

Multiply the two factors in the radicand together and combine the like terms.

$$\int \frac{2}{x} + 6 - \frac{4}{x^2} - \frac{12}{x}dx = \int 6 - \frac{10}{x} - \frac{4}{x^2}dx$$

Next use Rule 4 to integrate each term separately.

$$\int 6dx - 10\int \frac{1}{x}dx - 4\int x^{-2}dx$$

$$6x - 10\ln|x| - 4\frac{x^{-1}}{-1} + C$$

$$6x - 10\ln|x| + \frac{4}{x} + C$$

Differential Equations

A differential equation is one which has a derivative expression in it.

For example: f'(x)=2x+1 is a differential equation. When we integrate this type of equation we get the *general solution* which contains the constant, C .

To find a *particular solution* we need another piece of information. It could be a point that we know the function passes through. We call this piece of information the *initial condition*.

A typical problem might be:

$\left.\begin{array}{l} f'(x) = 3x^2 - 4x + 8 \\ f(1) = 9 \end{array}\right\}$

Let's look at the solution to the problem:

$\left.\begin{array}{l} f'(x) = 3x^2 - 4x + 8 \\ f(1) = 9 \end{array}\right\}$

Solution: First integrate both sides:

$$f(x) = \frac{3x^3}{3} - \frac{4x^2}{2} + 8x + C$$

Simplify: $f(x) = x^3 - 2x^2 + 8x + C$

Now find C by using the initial condition.
Substitute 1 for x and 9 for f(x)

$$9 = (1)^3 - 2(1)^2 + 8(1) + C$$
$$9 = 1 - 2 + 8 + C$$
$$9 = 7 + C$$
$$2 = C$$

This gives the particular solution.

$$f(x) = x^3 - 2x^2 + 8x + 2$$

Review - Basic Integration Rules

Rule 1: $\int kdx = kx + C$ (k, a constant)

Rule 2: The Power Rule $\int x^n = \frac{x^{n+1}}{n+1} + C$

Rule 3: The Indefinite Integral of a Constant Multiple of a Function

$$\int cf(x)dx = c\int f(x)dx, \text{ c is a constant}$$

Rule 4: The Sum Rule (or difference)

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$
$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

Rule 5: $\int e^x dx = e^x + C$

Rule 6: $\int \frac{1}{x}dx = \ln|x| + C$

Integration Formulas for Trigonometric Functions

$$\int \sin x\, dx = -\cos x + C$$
$$\int \cos x\, dx = \sin x + C$$
$$\int \sec^2 x\, dx = \tan x + C$$
$$\int \csc^2 x\, dx = -\cot x + C$$
$$\int \sec x \tan x\, dx = \sec x + C$$
$$\int \csc x \cot x\, dx = -\csc x + C$$

Examples.

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(a) $\int (3\csc x \cot x - 7\sec^2 x)dx$

(b) $\int \frac{3\tan\theta - 4\cos^2\theta}{\cos\theta} d\theta$

Solution to Example (a).

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$$\int (3\csc x \cot x - 7\sec^2 x)dx$$
$$= 3\int \csc x \cot x dx - 7\int \sec^2 x dx$$
$$= 3\left[-\csc x + C_1\right] - 7\left[\tan x + C_2\right]$$
$$= \underline{\underline{-3\csc x - 7\tan x + C}}$$

Solution to Example (b)

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$$\int \frac{3\tan\theta - 4\cos^2\theta}{\cos\theta} d\theta$$
$$= 3\int \frac{1}{\cos\theta}\tan\theta d\theta - 4\int \cos\theta d\theta$$
$$= 3\int \sec\theta \tan\theta d\theta - 4\int \cos\theta d\theta$$
$$= \underline{\underline{3\sec\theta - 4\sin\theta + C}}$$

Exercises:

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1. $\int y^3(2y^2 - 3)dy$
2. $\int \sqrt{x}(x+1)dx$
3. $\int \frac{y^4 + 2y^2 - 1}{\sqrt{y}} dy$
4. $\int \frac{\sin x}{\cos^2 x} dx$
5. $\int (2\cot^2\theta - 3\tan^2\theta)d\theta$

Applications of Indefinite Integrals

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1. Graphing

Given the sketch of the graph of the function, together with some function values, we can sketch the graph of its antiderivative as long as the antiderivative is continuous.

Example 1. Given the sketch of the function f =F′(x) below, sketch the possible graph of F if it is continuous, F(-1) = 0 and F(-3) = 4.

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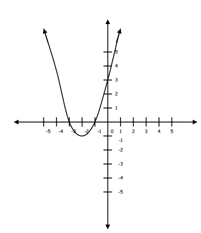

	F(x)	F′(x)	F″(x)	Conclusion
X<-3		+	-	Increasing, Concave down
X=-3	4	0	-	Relative maximum
-3<x<-2		-	-	Decreasing, Concave down
X=-2		-	0	Decreasing, Point of inflection
-2<x<-1		-	+	Decreasing Concave up
X=-1	0	0	+	Relative minimum
X>-1		+	+	Increasing, Concave up

The graph of F(x)

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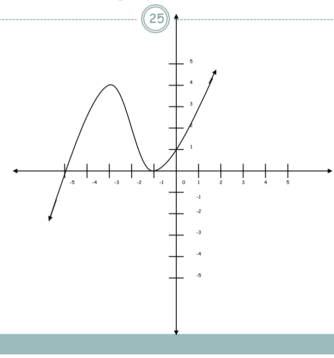

Applications of Indefinite Integrals

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1. Boundary/Initial Valued Problems

There are many applications of indefinite integrals in different fields such as physics, business, economics, biology, etc.

These applications usually desire to find particular antiderivatives that satisfies certain conditions called initial or boundary conditions, depending on whether they occur on one or more than one point.

Example 2.

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Suppose we wish to find a particular antiderivative satisfying the equation

$$\frac{dy}{dx} = 6x + 1$$

and the initial condition *y = 7* when *x = 2.*

Sol'n of Example 2

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$$dy = (6x+1)dx$$

$$\int dy = \int (6x+1)dx$$

$$y = 3x^2 + x + C$$

$$but \quad x = 2 \quad when \quad y = 7, \quad then$$

$$7 = 3(2)^2 + 3 + C \Rightarrow C = -7$$

Thus the particular antiderivative desired,

$$y = 3x^2 + x - 7$$

Example 3.

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The volume of water in a tank is *V* cubic meters when the depth of water is h meters. The rate of change of *V* with respect to *h* is $\pi(4h^2 + 12h + 9)$, find the volume of water in the tank when the depth is *3m.*

Sol'n of Example 3

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$$\frac{dV}{dh} = \pi\left(4h^2 + 12h + 9\right)$$

$$\int dV = \pi \int \left(4h^2 + 12h + 9\right)dh$$

$$V = \pi\left(\frac{4h^3}{3} + 6h^2 + 9h\right) + C$$

Volume V=0 if depth h =0

$$0 = \pi\left(\frac{4(0^3)}{3} + 6(0^2) + 9(0)\right) + C$$

$$\Rightarrow \quad C = 0$$

$$Thus \quad V = \pi\left(\frac{4h^3}{3} + 6h^2 + 9h\right)$$

$$V = \pi\left(\frac{4(3^3)}{3} + 6(3^2) + 9(3)\right) = 207\pi \quad m^3$$

The Differential Equations

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Equation containing a function and its derivative or just its derivative is called **differential equations.**

Applications occur in many diverse fields such as physics, chemistry, biology, psychology, sociology, business, economics etc.

The **order** of a differential equation is the order of the derivative of highest order that appears in the equation.

The function *f* defined by *y= f(x)* is a solution of a differential equation if *y* and its derivatives satisfy the equation.

The Differential Equations

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If each side of the differential equations involves only one variable or can be reduced in this form, then, we say that these are **separable differential equations.**

Complete solution (or general solution)

y = F(x) + C

Particular solution – an initial condition is given

Example 4.

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Find the complete solution of the differential equation

$$\frac{d^2y}{dx^2} = 4x + 3$$

Solution to Example 4.

Find the complete solution of the differential equation $\frac{d^2y}{dx^2} = 4x + 3$

$$\boxed{\begin{array}{l}\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ let \quad y' = \frac{dy}{dx}\end{array}}$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = 4x + 3$$

$$dy' = (4x+3)dx$$

$$\int dy' = \int (4x+3)dx$$

$$y' = 2x^2 + 3x + C_1$$

$$\frac{dy}{dx} = 2x^2 + 3x + C_1$$

$$\int dy = \int (2x^2 + 3x + C_1)dx$$

$$\underline{\underline{y = \tfrac{2}{3}x^3 + \tfrac{3}{2}x^2 + C_1x + C_2}}$$

Example 5.

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- Find the particular solution of the differential equation below for which y=2 and y'=-3 when x=1.

$$\frac{d^2y}{dx^2} = 4x + 3$$

Solution to Example 5.

Find the particular solution of the differential equation in Ex. 14 for which y=2 and y'=-3 when x=1.

$$y' = 2x^2 + 3x + C_1$$

$$-3 = 2(1)^2 + 3x + C_1$$

$$\Rightarrow C_1 = -8$$

$$y = \tfrac{2}{3}x^3 + \tfrac{3}{2}x^2 + C_1x + C_2$$

$$2 = \tfrac{2}{3}(1)^3 + \tfrac{3}{2}(1)^2 - 8(1) + C_2$$

$$\Rightarrow C_2 = \tfrac{47}{6}$$

$$\underline{\underline{y = \tfrac{2}{3}x^3 + \tfrac{3}{2}x^2 - 8x + \tfrac{47}{6}}}$$

Example 6.

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A stone is thrown vertically upward from the ground with an initial velocity of 20ft/sec.

(a) How long will the ball be going up?

(b) How high will the ball go?

(c) With what velocity will the ball strike the ground?

ANS to Example 6.

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A stone is thrown vertically upward from the ground with an initial velocity of 20ft/sec.

(a) How long will the ball be going up?
Ans. 0.625 sec

(b) How high will the ball go?
Ans. 6.25 ft

(c) With what velocity will the ball strike the ground?
Ans. 20 ft/sec

Other Properties of Indefinite Integrals

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- And two other interesting "properties" need to be highlighted:
- Interpret what the following 2 statement mean:
- Let $f(x) = x^3 - 2x$
- What is the answer for $\int f`(x)dx$ ....?
- What is the answer for $d/dx \int f(x)dx$ ..... ?

Examples with Motion

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- An object moves along a co-ordinate line with a velocity $v(t) = 2 - 3t + t^2$ meters/sec. Its initial position is 2 m to the right of the origin.
- (a) Determine the position of the object 4 seconds later
- (b) Determine the total distance traveled in the first 4 seconds

Examples – "B" Levels

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- Sometimes, the product rule for differentiation can be used to find an antiderivative that is not obvious by inspection
- So, by differentiating $y = x\ln x$, find an antiderivative for $y = \ln x$
- Repeat for $y = xe^x$ and $y = x\sin x$

Integration by Chain Rule/Substitution

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For integrable functions *f* and *g*

$$\int f(g(x))[g'(x)dx] = F(g(x)) + C$$

where is an *F* antiderivative of *f* and *C* is an arbitrary constant.

Example 1.

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$$\int 36x^2\sqrt[4]{6x^3+5}\,dx$$
$$= 2\int\sqrt[4]{6x^3+5}\,(18x^2dx)$$
$$= 2\int(6x^3+5)^{1/4}\,(18x^2dx)$$
$$= 2\frac{(6x^3+5)^{5/4}}{5/4}+C$$
$$= \tfrac{8}{5}(6x^3+5)^{5/4}+C$$

Let $g(x) = 6x^3+5$
$g'(x)=18x^2$

Example 1. Take 2!

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$$\int 36x^2\sqrt[4]{6x^3+5}\,dx = 2\int\sqrt[4]{6x^3+5}\,(18x^2dx)$$
$$= 2\int\sqrt[4]{u}\,du$$
$$= 2\int(u)^{1/4}\,du$$
$$= 2\frac{u^{5/4}}{5/4}+C$$
$$= \tfrac{8}{5}u^{5/4}+C$$
$$= \tfrac{8}{5}(6x^3+5)^{5/4}+C$$

Let $u = 6x^3 + 5$
$du = 18x^2\,dx$

Example 2.

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Let $g(t) = t^4 + 2t$
$g'(t) = 4t^3 + 2$
$= 2(2t^3 + 1)$

$$\int\frac{2t^3+1}{\left(t^4+2t\right)^7}dx = \frac{1}{2}\int\frac{2(2t^3+1)dx}{\left(t^4+2t\right)^7}$$
$$= \frac{1}{2}\int\left(t^4+2t\right)^{-7}\left[2(2t^3+1)dx\right]$$
$$= \frac{1}{2}\frac{\left(t^4+2t\right)^{-6}}{-6}+C$$
$$= \frac{-1}{12\left(t^4+2t\right)^6}+C$$

Example 3.

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$$\int x^5(x^2-1)^{12}\,2dx$$
$$= \int\left(x^2\right)^2(x^2-1)^{12}\,2xdx$$
$$= \int(u+1)^2u^{12}\,du$$
$$= \int(u^{14}+2u^{13}+u^{12})du$$
$$= \tfrac{1}{15}u^{15}+\tfrac{2}{14}u^{14}+\tfrac{1}{13}u^{13}+C$$
$$= \tfrac{1}{15}(x^2-1)^{15}+\tfrac{1}{7}(x^2-1)^{14}+\tfrac{1}{13}(x^2-1)^{13}+C$$

Let $u = x^2 - 1$
$du = 2x\,dx$
$x^2 = u+1$

Example 4.

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$$\int\sin 2x\sqrt{2-\cos 2x}dx$$
$$= \tfrac{1}{2}\int\sqrt{2-\cos 2x}\,(2\sin 2xdx)$$
$$= \tfrac{1}{2}\int\sqrt{u}\,(du)$$
$$= \tfrac{1}{2}\int u^{1/2}(du)$$
$$= \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2}\right)+C$$
$$= \tfrac{1}{3}(2-\cos 2x)^{3/2}+C$$

Let $u = 2 - \cos 2x$
$du = 0 - (-\sin 2x)(2dx)$
$= 2\sin 2x dx$

Example 5.

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$$\int(\tan 2x+\cot 2x)^2dx$$
$$= \int\left(\frac{\sin 2x}{\cos 2x}+\frac{\cos 2x}{\sin 2x}\right)^2dx$$
$$= \int\left(\frac{\sin^2 2x+\cos^2 2x}{\cos 2x\sin 2x}\right)^2dx$$
$$= \int\frac{1}{\cos^2 2x\sin^2 2x}dx$$
$$= \int\sec^2 2x\csc^2 2xdx$$

Example 6

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$$= \int \sec^2 2x \csc^2 2x dx$$
$$= \int \sec^2 2x(\cot^2 2x + 1)dx$$
$$= \int \sec^2 2x \cot^2 2x dx + \int \sec^2 2x dx$$
$$= \int (\tan 2x)^{-2} \sec^2 2x dx + \int \sec^2 2x dx$$
$$= \tfrac{1}{2}\int (\tan 2x)^{-2} 2\sec^2 2x dx + \tfrac{1}{2}\int 2\sec^2 2x dx$$
$$= -\tfrac{1}{2}(\tan 2x)^{-1} + \tfrac{1}{2}\tan 2x + C$$
$$= -\tfrac{1}{2}\cot 2x + \tfrac{1}{2}\tan 2x + C$$

Exercises:

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1. $\int 7x(2x^2+1)^6 dx$
2. $\int 5x\sqrt[3]{(9-4x^2)^2}$
3. $\int \frac{2r}{(1-r)^7} dr$
4. $\int y \csc 3y^2 \cot 3y^2 dy$
5. $\int \frac{\cos 3x}{\sqrt{1-2\sin 3x}} dx$
6. $\int \frac{x(3x^2+1)dx}{(3x^4+2x^2+1)^4}$