Lesson 36 – Derivatives of Composed Functions – The Chain Rule

Calculus - Santowski

Fast Five

- Given the following expressions for f(x) and g(x), find the equation for fog(x)

$f(x)=x^2$ & $g(x)=x^3-x+1$

$f(x)=\frac{1}{x+2}$ & $g(x)=\sqrt{x-3}$

$f(x)=\sin(x)$ & $g(x)=e^{2x}$

$f(x)=\ln\left(x^2+\sin(x)\right)$ & $g(x)=\frac{1}{x+1}$

- Given the following composed functions, decompose them into f(x) and g(x)

$f \circ g(x)=\sqrt{x^2-4}$

$f \circ g(x)=\frac{1}{\sqrt{x}}$

$f \circ g(x)=(3x+2)^{-3}$

$f \circ g(x)=2\sin\left(\sqrt{x^2-2}\right)$

$f \circ g(x)=e^{x^2-x-2}$

$f \circ g(x)=\ln(\tan(x))$

Lesson Objectives

- Investigate patterns in the derivatives of composed functions
- Apply the Chain Rule to differentiate composite functions
- Apply the Chain Rule in the analysis of functions
- Apply the Chain Rule in mathematical modeling

(A) Exploration

- You are given a worksheet and we will work through the first row together so that you understand what I require of you:
- Decompose $C(x)=f\circ g(x)=\left(x^2+4\right)^2$
- And tell me what f(x) = and what g(x) =
- Now work out the individual derivatives of f(x) and g(x)
- Now use WolframAlpha to find the derivative of C(x) or d/dx fog(x)
- Repeat and look PATTERNS

(A) Exploration

- Now continue with $C(x)=f\circ g(x)=\left(x^2+4\right)^3$
- Now use WolframAlpha to find the derivative of C(x) or d/dx fog(x)
- Repeat with $C(x)=f\circ g(x)=\left(x^2+4\right)^4$
- Repeat with $C(x)=f\circ g(x)=\left(x^2+4\right)^5$
- Repeat and look PATTERNS

(A) Exploration

- Now revisit $C(x)=f\circ g(x)=\left(x^2+4\right)^2$
- Now use WolframAlpha to find the derivative of C(x) or d/dx fog(x)
- Repeat with $C(x)=f\circ g(x)=\left(x^3+4\right)^2$
- Repeat with $C(x)=f\circ g(x)=\left(x^4+4\right)^2$
- Repeat with $C(x)=f\circ g(x)=\left(x^2+4x\right)^2$
- Repeat with $C(x)=f\circ g(x)=\left(2x^3+4x^2\right)^2$
- Repeat and look PATTERNS

(B) The Chain Rule – Function Notation

- The chain rule presents a formula that we can use to take the derivatives of these compose functions.
- (i) In function notation, we can write the chain rule as follows:
- Given that *f* and *g* are differentiable and $F = f \circ g$ is the composed function defined by $F = f(g(x))$, then $F`(x)$ is given by the product $F`(x) = f`(g(x)) \times g`(x)$.
- We can try to understand composite functions and their derivatives in the following manner:
- $f(g(x))$ => means that *f* is the outer function into which we have substituted an inner function of *g*. So the derivative is then the product of the derivative of the outer function, *f*, evaluated at the inner function times the derivative of the inner function.

(C) The Chain Rule – Leibniz Notation

- (ii) In Leibniz notation, we can write the Chain rule as follows:
- If $y = f(u)$, where $u = g(x)$ and *f* and *g* are differentiable, then *y* is a differentiable function of *x* and dy/dx = dy/du x du/dx
- We can try to understand the formula using this example:
- If we have the composed function $f(x) = (2x^2 + 3)^{½}$, then we could "decompose" the function into $y = f(u) = (u)^{½}$ where $u(x) = 2x^2 + 3$.
- So if we want the derivative of $f(u) = (u)^{½}$, then we can understand that the variable in $y = f(u)$ is *u*, so we can only take the derivative of *y* with respect to *u*, hence the idea of dy/du.
- However, we were asked for the derivative of the function *F* with respect to *x*, so we then simply "follow up" the derivative of dy/du by differentiating *u* with its variable of *x*, hence the idea of du/dx

(D) The Chain Rule

- Find dy/dx if $y = (2x^2 + 3)^{½}$
- Let $u(x) = 2x^2 + 3$ and then $y(u) = u^{½}$
- The derivative formula is dy/dx = dy/du x du/dx → so …
- If $y(u) = u^{½}$, then dy/du = $½\, u^{-½}$
- Then if $u(x) = 2x^2 + 3$, then du/dx = $4x$ → so …
- If we put it all together → dy/dx = dy/du x du/dx → we get
- dy/dx = $(½\, u^{-½}) \times (4x)$ and then dy/dx = $[½(2x^2 + 3)^{-½}] \times (4x)$
- So then dy/dx = $2x(2x^2 + 3)^{-½}$

(E) Chain Rule - Summary

- We can understand the chain rule in two notations:
- (i) $\frac{d}{dx}\; f \circ g(x) = f'(g(x)) \times g'(x)$
- (ii) if y is a function of u $\Rightarrow$ $y = f(u)$:

  then $f'(x) = \frac{dy}{du} \times \frac{du}{dx}$

(F) Examples

1) $y = (x^3 + 3)^5$
2) $y = (-3x^5 + 1)^3$
3) $y = (-5x^3 - 3)^3$
4) $y = (5x^2 + 3)^4$
5) $f(x) = \sqrt[4]{-3x^4 - 2}$
6) $f(x) = \sqrt{-2x^2 + 1}$
7) $f(x) = \sqrt[3]{-2x^4 + 5}$
8) $y = (-x^4 - 3)^{-2}$

(F) Example

- Differentiate and simplify

$$h(t) = \left(\frac{2t-1}{t+2}\right)^{-3}$$

$$f(t) = \ln(x^3 + 4x)$$

$$g(t) = x^2 e^{x^2}$$

- Find the extreme value(s) of $y = \sqrt{4x - \frac{1}{2}x^2}$

(G) Examples

- 1. Find the equation of the tangent line at x = 2 to the curve $g(x)=\frac{1}{\sqrt{20-x^4}}$
- 2. Given the function $g(x)=e^{x+x^2}$
- (a)find the location(s) of the horizontal tangent lines to the given curve
- (b) Determine the interval(s) of decrease
- (c) Determine the eqn of the tangent line at x = 0

(H) Examples

- (a) Where is $f(x) = (x^2 - x - 2)^5$ increasing?
- (b) Find and classify the extrema on

$$(a)\;\; h(t)=\left(\sqrt{2t-2}\right)(2t+5)^{-1}$$
$$(b)\;\; g(t)=\ln(x^2+x^{-2})$$