Lesson 33 – Derivatives of Power Functions

Calculus – Santowski

POP QUIZ

- Sketch a function such that the following criteria are satisfied by function:

$$(a)\ \lim_{x\to 0^+} f(x) = -2 \qquad (b)\ \lim_{x\to 0^-} f(x) = 1 \qquad (c)\ f(0)=1$$
$$(d)\ \lim_{x\to 2^-} f(x) = \infty \qquad (e)\ \lim_{x\to 2^+} f(x) = -\infty$$
$$(f)\ \lim_{x\to \infty} f(x) = 3 \qquad (g)\ \lim_{x\to -\infty} f(x) = 4$$

Lesson Objectives

- 1. Use first principles (limit definitions) to develop the power rule
- 2. Use graphic differentiation to verify the power rule
- 3. Use graphic evidence to verify antiderivative functions
- 4. Apply the power rule to real world problems
- 5. Apply the power rule to determine characteristics of polynomial function

Fast Five

- 1. Factor the following (use Wolframalpha if you get stuck)
  - $x^3 - 8$ $\quad x^3 - 27$
  - $x^5 - 32$ $\quad x^7 - 128$
  - $x^{11} - 2048$ $\quad x^6 - 2^6$
- 2. Given your factorizations in Q1, predict the factorization of $x^n - a^n$
- 3. Given $f(x) = x^n$ & your previous work, evaluate

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

(A) Review

- The equation used to find the slope of a tangent line or an instantaneous rate of change is:

$$m = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

- which we also then called a derivative.
- So derivatives are calculated as .

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x) = \frac{dy}{dx}$$

(B) Finding Derivatives – Graphical Investigation

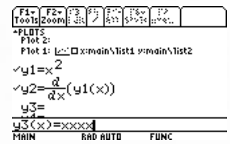
- We will now develop a variety of useful differentiation rules that will allow us to calculate equations of derivative functions much more quickly (compared to using limit calculations each time)
- First, we will work with simple power functions
- We shall investigate the derivative rules by means of the following algebraic and GC investigation (rather than a purely "algebraic" proof)

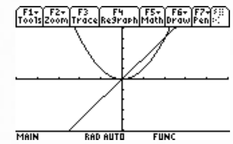
(B) Finding Derivatives – Graphical Investigation

- Use your GDC to graph the following functions (each in y1(x)) and then in y2(x) graph d(y1(x),x)
- Then in y3(x) you will enter an equation that you think overlaps the derivative graph from y2(x)

- (1) d/dx (x^2) (2) d/dx (x^3)
- (3) d/dx (x^4) (4) d/dx (x^5)
- (5) d/dx (x^{-2}) (6) d/dx (x^{-3})
- (7) d/dx ($x^{0.5}$) (8) d/dx (x)

(B) Finding Derivatives – Graphical Investigation





- As an example, as you investigate y = x2, you will enter an equation into y3(x) ..... If it doesn't overlap the derivative graph from y2(x), try again until you get an overlap

(B) Finding Derivatives – Graphical Investigation

- Conclusion to your graphical investigation:

- (1) d/dx (x^2) = 2x (2) d/dx (x^3) = $3x^2$
- (3) d/dx (x^4) = $4x^3$ (4) d/dx (x^5) = $5x^4$
- (5) d/dx(x^{-2}) = $-2x^{-3}$ (6) d/dx (x^{-3}) = $-3x^{-4}$
- (7) d/dx ($x^{0.5}$) = $0.5x^{-0.5}$ (8) d/dx (x) = 1

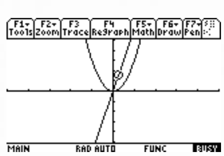
- Which suggests a generalization for f(x) = x^n
- The derivative of x^n ==> nx^{n-1} which will hold true for all n

(C) Finding Derivatives - Sum and Difference and Constant Rules

- Now that we have seen the derivatives of power functions, what about functions that are made of various combinations of power functions (i.e. sums and difference and constants with power functions?)

- Ex 1: d/dx ($3x^2$) d/dx($-4x^{-2}$)
- Ex 2: d/dx($x^2 + x^3$) d/dx($x^{-3} + x^{-5}$)
- Ex 3: d/dx (x^4 - x) d/dx ($x^3 - x^{-2}$)

(C) Finding Derivatives - Sum and Difference and Constant Rules



- Use the same graphical investigation approach:

- Ex 1: d/dx ($3x^2$) = ? d/dx($-4x^{-2}$) = ?
- Ex 2: d/dx($x^2 + x^3$) = ? d/dx($x^{-3} + x^{-5}$) = ?
- Ex 3: d/dx (x^4 - x) = ? d/dx ($x^3 - x^{-2}$) = ?

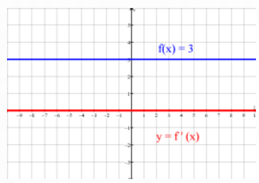
(C) Finding Derivatives - Sum and Difference and Constant Rules

- The previous investigation leads to the following conclusions:

- (1) $\frac{d}{dx}\left(k \cdot x^n\right) = k \cdot \frac{d}{dx}\left(x^n\right) = k \cdot \left(nx^{n-1}\right) = knx^{n-1}$
- (2) $\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}\left(f(x)\right) + \frac{d}{dx}\left(g(x)\right)$
- (3) $\frac{d}{dx}\left(f(x) - g(x)\right) = \frac{d}{dx}\left(f(x)\right) - \frac{d}{dx}\left(g(x)\right)$

(C) Constant Functions

- (i) f(x) = 3 is called a constant function ➔ graph and see why.
- What would be the rate of change of this function at *x* = 6? *x* = -1, *x* = *a*?
- We could do a limit calculation to find the derivative value ➔ but we will graph it on the GC and graph its derivative.
- So the derivative function equation is f `(x) = 0



(D) Examples

- Ex 1: Differentiate the following:
- (a) $g(x) = 3x^4 - 3x^2 + 4 - \frac{1}{x}$
- (b) $b(x) = 0.1x^3 + 2x^{\frac{5}{2}} - x$
- Ex 2. Find the second derivative :
- (a) f(x) = x^2 (b) g(x) = x^3
- (c) h(x) = $x^{1/2}$

(D) Examples

- Ex 1: Differentiate the following:
- (a) $g(x) = 5\sqrt{x} - \frac{10}{x^2} + \frac{1}{2\sqrt{x}} - 3x^4 + 1$
- (b) $b(x) = 0.1x^3 + 2x^{\sqrt{2}} - \frac{2}{x^{\pi}}$
- Ex 2. Find the second derivative :
- (a) f(x) = x^2 (b) g(x) = x^3
- (c) h(x) = $x^{1/2}$

(E) Examples - Analyzing Functions

- Ex 1: Find the equation of the line which is normal/orthogonal to the curve y = x^2 - 2x + 4 at x = 3.
- Ex 2. Given the parabola y = $2x^2$ + 6x + 5, find the point at which the line y = -4x+b is tangent to the parabola. Hence, find the value of b.
- Ex 3. Given an external point A(-4,0) and a parabola f(x) = x^2 - 2x + 4, find the equations of the 2 tangents to f(x) that pass through A

(E) Examples - Analyzing Functions

- Ex 4: At which x-values does the function f(x) = x^4 - $4x^3$ have extrema?
- Ex 5: Determine the co-ordinates of the extrema of the function g(x) = x^5 - 16x?

(F) Examples - Applications

- A ball is dropped from the top of the Empire State building to the ground below. The height in feet, h(t), of the ball above the ground is given as a function of time, t, in seconds since release by h(t) = 1250 - $16t^2$
- (a) Determine the velocity of the ball 5 seconds after release
- (b) How fast is the ball going when it hits the ground?
- (c) what is the acceleration of the ball?

(G) Examples - Economics

- Suppose that the total cost in hundreds of dollars of producing x thousands of barrels of oil is given by the function $C(x) = 4x^2 + 100x + 500$. Determine the following.

- (a) the cost of producing 5000 barrels of oil
- (b) the cost of producing 5001 barrels of oil
- (c) the cost of producing the 5001st barrelof oil
- (d) C `(5000) = the marginal cost at a production level of 5000 barrels of oil. Interpret.
- (e) The production level that minimizes the average cost (where AC(x) = C(x)/x))

(G) Examples - Economics

- Revenue functions:

- A demand function, p = f(x), relates the number of units of an item that consumers are willing to buy and the price of the item
- Therefore, the revenue of selling these items is then determined by the amount of items sold, x, and the demand (# of items)
- Thus, R(x) = xp(x)

(G) Examples - Economics

- The demand function for a certain product is given by p(x) = (50,000 - x)/20,000

- (a) Determine the marginal revenue when the production level is 15,000 units.
- (b) If the cost function is given by C(x) = 2100 - 0.25x, determine the marginal profit at the same production level
- (c) How many items should be produced to maximize profits?