Lesson 31 (Day 2) - Limits

Calculus - Mr Santowski

Lesson Objectives

- 1. Define limits
- 2. Use algebraic, graphic and numeric (AGN) methods to determine if a limit exists
- 3. Use algebraic, graphic and numeric methods to determine the value of a limit, if it exists
- 4. Use algebraic, graphic and numeric methods to determine the value of a limit at infinity, if it exists
- 5. Be able to state and then work with the various laws of limits
- 6. Apply limits to application/real world problems

Fast Five - Limits (Lesson #2)

- Consider the following limit(s):

$$\lim_{x\to\infty} f(x) \quad if \quad f(x) = \frac{1}{x^n}$$

$$\lim_{x\to 0} f(x) \quad if \quad f(x) = \frac{1}{x^n}$$

- Determine the limiting value(s) if possible.

(A) Introduction to Limits

- Let *f* be a function and let *a* and *L* be real numbers. If
- 1. As *x* takes on values closer and closer (but not equal) to *a* on both sides of *a*, the corresponding values of *f*(*x*) get closer and closer (and perhaps equal) to *L*; and
- 2. The value of *f*(*x*) can be made as close to *L* as desired by taking values of *x* close enough to *a*;
- Then *L* is the LIMIT of *f*(*x*) as *x* approaches *a*
- Written as $\lim_{x\to a} f(x) = L$

(C) Existence of Limits

- Now here is an equation of a function which is defined as

$$f(x) = \begin{cases} 2-x & x<2 \\ (x-3)^2-2 & x\geq 2 \end{cases}$$

- Find $\lim_{x\to 2} f(x)$

- Here is an equation of a function which is defined as

$$f(x) = \frac{|x-2|}{x-2}$$

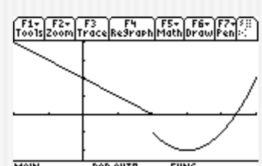
- Find $\lim_{x\to 2} f(x)$

(C) Existence of Limits

- Now here is a graph of a function which is defined as

$$f(x) = \begin{cases} 2-x & x<2 \\ (x-3)^2-2 & x\geq 2 \end{cases}$$

- Find $\lim_{x\to 2} f(x)$



(C) Existence of Limits

- In considering several of our previous examples, we see the idea of one and two sided limits.
- A **one sided limit** can be a **left handed limit** notated as $\lim_{x\to a^-} f(x)$ which means we approach *x* = *a* from the left (or negative) side
- We also have **right handed limits** which are notated as $\lim_{x\to a^+} f(x)$ which means we approach *x* = *a* from the right (or positive) side

(C) Existence of Limits

- We can make use of the left and right handed limits and now define conditions under which we say a function does not have a limiting y value at a given x value ==> by again considering our various examples above, we can see that some of our functions do not have a limiting y value because as we approach the x value from the right and from the left, we do not reach the same limiting y value.
- Therefore, if $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$ then $\lim_{x\to a} f(x)$ does not exist.

(C) Existence of Limits

- Now here is an equation of a function which is defined as

$$f(x)=\begin{cases} 3-x & x<2 \\ \dfrac{2x}{x+2} & x\geq 2 \end{cases}$$

- Find lim $_{x\to 2}$ f(x)
- Here is an equation of a function which is defined as

$$f(x)=\frac{x^2-x-6}{x-3}$$

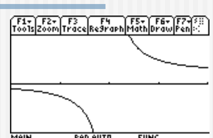
- Find lim $_{x\to 3}$ f(x)

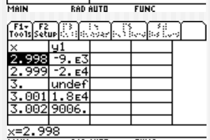
(C) Existence of Limits

- We can make use of the left and right handed limits and now define conditions under which we say a function does have a limiting y value at a given x value ==> by again considering our various examples above, we can see that some of our functions do have a limiting y value because as we approach the x value from the right and from the left, we reach the same limiting y value.
- Therefore, if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ then $\lim_{x\to a} f(x)$ does exist.

(D) Limits & Infinities

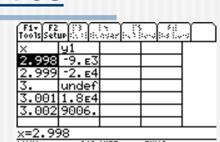
- Find $\lim_{x\to 3} \dfrac{x^2+9}{x-3}$
- So we try to use some algebra "tricks" as before, but $x^2 + 9$ doesn't factor.
- So we use a ToV, and a graph
- What is the limit in this case?

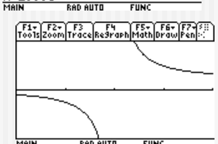




(D) Limits & Infinities

- In this case, both the graph and the table suggest two things:
- (1) as x→3 from the left, g(x) becomes more and more negative
- (2) as x→3 from the right, g(x) becomes more and more positive





(D) Limits & Infinities

- So we write these ideas as:
- $\lim_{x\to 3^-} \frac{x^2+9}{x-3} = -\infty$ & $\lim_{x\to 3^+} \frac{x^2+9}{x-3} = \infty$
- Since there is no real number that g(x) approaches, we simply say that this limit does not exist

(D) Limits at Infinity

- Consider the following limit(s):

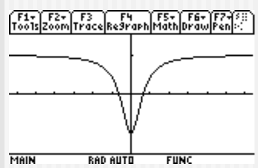
$$\lim_{x\to\infty} f(x) \quad if \quad f(x) = \frac{1}{x^n}$$

- Determine the limiting value(s) if possible.

(D) Limits at Infinity

- In considering limits at infinity, we are being asked to make our x values infinitely large and thereby consider the "end behaviour" of a function
- Consider the limit $\lim_{x\to\pm\infty} \frac{x^2-1}{x^2+1}$ numerically, graphically and algebraically
- We can generate a table of values and a graph (see next slide)
- So here the function approaches a limiting value, as we make our x values sufficiently large ➔ we see that f(x) approaches a limiting value of 1 ➔ in other words, a horizontal asymptote

(D) Limits at Infinity – Graph & Table



x	y6
0.	-1.
50.	.9992
100.	.9998
150.	.9999
200.	1.

y6(x)=.99995000124997

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(D) Limits at Infinity – Algebra

- $$\lim_{x\to+\infty}\left(\frac{x^2-1}{x^2+1}\right) = \lim_{x\to+\infty}\left(\frac{\frac{x^2}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}}\right) = \lim_{x\to+\infty}\left(\frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}\right) = \frac{1-0}{1+0} = 1$$

- Divide through by the highest power of *x*
- Simplify
- Substitute x = ∞ ➔ 1/∞ → 0

(D) Examples of Limits at Infinity

- Work through the following examples graphically, numerically or algebraically
- (i) $\lim_{x\to\infty}\left(\frac{3x^2-x-2}{-5x^2+4x+1}\right)$
- (ii) $\lim_{x\to\infty}\left(\frac{3x^4-x-2}{-5x^2+4x+1}\right)$
- (iii) $\lim_{x\to\infty}\left(\frac{3x^2-x-2}{-5x^4+4x+1}\right)$

- Work through the following examples graphically, numerically or algebraically

$$\lim_{x\to-\infty}\left(\tan^{-1}(x)\right)$$

$$\lim_{x\to\infty}\left(\sqrt{x^2+2}-x\right)$$

(E) Limit Laws

- The limit of a constant function is the constant
- The limit of a sum is the sum of the limits
- The limit of a difference is the difference of the limits
- The limit of a constant times a function is the constant times the limit of the function
- The limit of a product is the product of the limits
- The limit of a quotient is the quotient of the limits (if the limit of the denominator is not 0)
- The limit of a power is the power of the limit
- The limit of a root is the root of the limit

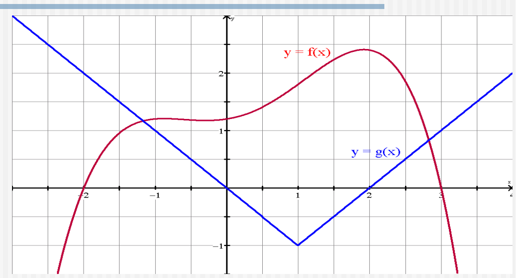
(E) Limit Laws

- Here is a summary of some important limits laws:
- (a) sum/difference rule ➔ $\lim [f(x) \pm g(x)] = \lim f(x) \pm \lim g(x)$
- (b) product rule ➔ $\lim [f(x) \times g(x)] = \lim f(x) \times \lim g(x)$
- (c) quotient rule ➔ $\lim [f(x) \div g(x)] = \lim f(x) \div \lim g(x)$
- (d) constant multiple rule ➔ $\lim [kf(x)] = k \times \lim f(x)$
- (e) constant rule ➔ $\lim (k) = k$
- These limits laws are easy to work with, especially when we have rather straight forward polynomial functions

(E) Limit Laws - Examples

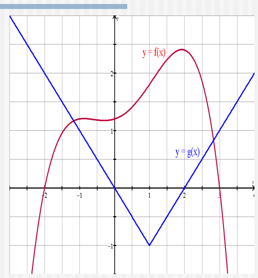
- Find $\lim_{x\to 2} (3x^3 - 4x^2 + 11x - 5)$ using the limit laws
- $\lim_{x\to 2} (3x^3 - 4x^2 + 11x - 5)$
- $= 3 \lim_{x\to 2} (x^3) - 4 \lim_{x\to 2} (x^2) + 11 \lim_{x\to 2} (x) - \lim_{x\to 2} (5)$
- $= 3(8) - 4(4) + 11(2) - 5$ (using simple substitution or use GDC)
- $= 25$
- For the rational function f(x), find
- $\lim_{x\to 2} (2x^2 - x) / (0.5x^3 - x^2 + 1)$
- $= [2 \lim_{x\to 2} (x^2) - \lim_{x\to 2} (x)] / [0.5 \lim_{x\to 2} (x^3) - \lim_{x\to 2} (x^2) + \lim_{x\to 2} (1)]$
- $= (8 - 2) / (4 - 4 + 1)$
- $= 6$

(E) Limit Laws and Graphs



(E) Limit Laws and Graphs

- From the graph on this or the previous page, determine the following limits:
- (1) $\lim_{x\to -2} [f(x) + g(x)]$
- (2) $\lim_{x\to -2} [(f(x))^2 - g(x)]$
- (3) $\lim_{x\to -2} [f(x) \times g(x)]$
- (4) $\lim_{x\to -2} [f(x) \div g(x)]$
- (5) $\lim_{x\to 1} [f(x) + 5g(x)]$
- (6) $\lim_{x\to 1} [\frac{1}{2}f(x) \times (g(x))^3]$
- (7) $\lim_{x\to 2} [f(x) \div g(x)]$
- (8) $\lim_{x\to 2} [g(x) \div f(x)]$
- (9) $\lim_{x\to 3} [f(x) \div g(x)]$



(I) Exploration

- Research the DELTA-EPSILON definition of a limit
- Tell me what it is and be able to use it
- MAX 2 page hand written report (plus graphs plus algebra) + 2 Q quiz