## Lesson 31 - Limits

Calculus - Mr Santowski

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## Lesson Objectives

- 1. Define limits
- 2. Use algebraic, graphic and numeric (AGN) methods to determine if a limit exists
- 3. Use algebraic, graphic and numeric methods to determine the value of a limit, if it exists
- 4. Use algebraic, graphic and numeric methods to determine the value of a limit at infinity, if it exists
- 5. Be able to state and then work with the various laws of limits
- 6. Apply limits to application/real world problems 1/21/2015 Mr. Santowski - Calculus \& IBHL



## Fast Five - Limits and Graphs

- Find the limit of the function $f(x)$ at the following values:
- (i) the limit of $f(x)$ at $x=-10$ is
- (ii) the limit of $f(x)$ at $x=-6$ is
- (iii) the limit of $f(x)$ at $x=-4$ is
- (iv) the limit of $f(x)$ at $x=0$ is
- (v) the limit of $f(x)$ at $x=2$ is
- (vi) the limit of $f(x)$ at $x=4$ is
- (vii) the limit of $f(x)$ at $x=6$ is
- (i) the limit of $f(x)$ at $x=7$ is
- (i) the limit of $f(x)$ at $x=10$ is


## (A) Introduction to Limits

- Let $f$ be a function and let $a$ and $L$ be real numbers. If
- 1. As $x$ takes on values closer and closer (but not equal) to $a$ on both sides of $a$, the corresponding values of $f(x)$ get closer and closer (and perhaps equal) to $L$; and
- 2. The value of $f(x)$ can be made as close to $L$ as desired by taking values of $x$ close enough to $a$;
- Then $L$ is the LIMIT of $f(x)$ as $x$ approaches a
- Written as $\lim _{x \rightarrow a} f(x)=L$

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## (A) Introduction to Limits

- We will work with $g(x)=\frac{x^{3}-8}{x-2}$ and consider the function behaviour at $\mathrm{x}=2$
- We can express this idea of function behaviour at a point using limit notation as

$$
\lim _{x \rightarrow 2} \frac{x^{3}-8}{x-2}
$$

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## (A) Introduction to Limits

- We will explore the limit in a variety of ways: first using a ToV
- So notice what happens to the function values as $x$ gets closer to 2 from both sides (RS 2.01, 2.02 \& LS 1.98, 1.99)
- So we can predict a limiting function value of 12

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- We will explore the limit in a variety of ways: now using a graph and tracing the function values



## (B) Determining Values of Limits

- Now, how does all the algebra tie into limits?
- If we try a direct substitution to evaluate the limit value, we get $0 / 0$ which is indeterminate

$$
\begin{aligned}
& \lim _{x \rightarrow 2} g(x) \\
& =\lim _{x \rightarrow 2} \frac{x^{3}-8}{x-2} \\
& =\lim _{x \rightarrow 2} \frac{(2)^{3}-8}{(2)-2} \\
& =\frac{0}{0}
\end{aligned}
$$

## (B) Determining Values of Limits

- Is there some way that we can use our algebra skills to come to the same answer?
- Four skills become important initially: (1) factoring \& simplifying, (2) rationalizing and (3) common denominators and (4) basic function knowledge

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## (B) Determining Values of Limits

- Consider the expression $g(x)=\frac{x^{3}-8}{x-2}$
- Now can we factor a difference of cubes?
$g(x)=\frac{x^{3}-8}{x-2}$
$g(x)=\frac{(x-2)\left(x^{2}+2 x+4\right)}{(x-2)}$
$g(x)=x^{2}+2 x+4 \quad, x \neq 2$

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## (B) Determining Values of Limits

- Determine the following limits.

$$
\lim _{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3}
$$

- Each solution introduces a different "algebra" trick for simplifying the rational expressions
- Verify limit on GDC
$\lim _{x \rightarrow 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2}$
$\lim _{h \rightarrow 0} \frac{(4+h)^{3}-64}{h}$
$\lim _{h \rightarrow 0} \frac{\frac{1}{(2+h)^{2}}-\frac{1}{4}}{h}$

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## (C) Existence of Limits

- In this case, both the graph and the table suggest two things:
- (1) as $x \rightarrow 3$ from the left, $g(x)$ becomes more and more negative
- (2) as $x \rightarrow 3$ from the right, $g(x)$ becomes more and more positive



## (C) Existence of Limits

- So we write these ideas as:
- $\lim _{x \rightarrow 3^{-}} \frac{x^{2}+9}{x-3}=-\infty$
\& $\quad \lim _{x \rightarrow 3^{+}} \frac{x^{2}+9}{x-3}=\infty$
- Since there is no real number that $g(x)$ approaches, we simply say that this limit does not exist

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## (C) Existence of Limits

- Now find the limit of this function as $x$ approaches 2 where $f(x)$ is defined as

$$
f(x)=\left\{\begin{array}{cc}
4-3 x & x<2 \\
(2 x-3)^{2}-2 & x \geq 2
\end{array}\right.
$$

- i.e. determine $\lim _{x \rightarrow 2} f(x)$


## (C) Existence of Limits

- In considering several of our previous examples, we see the idea of one and two sided limits.
- A one sided limit can be a left handed limit notated as $\lim _{x \rightarrow a^{-}} f(x) \quad$ which means we approach $x=a$ from the left (or negative) side
- We also have right handed limits which are notated as $\lim _{x \rightarrow a^{+}} f(x) \quad$ which means we approach $x=a$ from the $\stackrel{x}{x \rightarrow a^{+}}$right (or positive) side

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## (C) Existence of Limits

- We can make use of the left and right handed limits and now define conditions under which we say a function does not have a limiting $y$ value at a given $x$ value $==>$ by again considering our various examples above, we can see that some of our functions do not have a limiting y value because as we approach the $x$ value from the right and from the left, we do not reach the same limiting y value.
- Therefore, if $\lim _{x \rightarrow a^{-}} f(x) \neq \lim _{x \rightarrow a^{+}} f(x)$ then $\lim _{x \rightarrow a} f(x)$ does not exist.

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## (C) Existence of Limits

- Now here is a graph of a function which is defined as

$$
f(x)=\left\{\begin{array}{cc}
2-x & x<2 \\
(x-3)^{2}-2 & x \geq 2
\end{array}\right.
$$

- Find $\lim _{x \rightarrow 2} f(x)$


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## (E) Limit Laws

- The limit of a constant function is the constant
- The limit of a sum is the sum of the limits
- The limit of a difference is the difference of the limits
- The limit of a constant times a function is the constant times the limit of the function
- The limit of a product is the product of the limits
- The limit of a quotient is the quotient of the limits (if the limit of the denominator is not 0 )
- The limit of a power is the power of the limit
- The limit of a root is the root of the limit
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## (E) Limit Laws

- Here is a summary of some important limits laws:
- (a) sum/difference rule $\rightarrow \lim [f(x) \pm g(x)]=\lim f(x) \pm \lim g(x)$
- (b) product rule $\rightarrow \lim [f(x) \times g(x)]=\lim f(x) \times \lim g(x)$
- (c) quotient rule $\rightarrow \lim [f(x) \div g(x)]=\lim f(x) \div \lim g(x)$
- (d) constant multiple rule $\rightarrow \lim [k f(x)]=k \times \lim f(x)$
- (e) constant rule $\rightarrow \lim (k)=k$
- These limits laws are easy to work with, especially when we have rather straight forward polynomial functions
(E) Limit Laws and Graphs



## (G) Internet Links

- Limit Properties - from Paul Dawkins at Lamar University
- Computing Limits - from Paul Dawkins at Lamar University
- Limits Theorems from Visual Calculus
- Exercises in Calculating Limits with solutions from UC Davis

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## (G) Internet Links

- Limits Involving Infinity from Paul Dawkins at Lamar University
- Limits Involving Infinity from Visual Calculus
- Limits at Infinity and Infinite Limits from Pheng Kim Ving
- Limits and Infinity from SOSMath


## (I) "A" Level Investigation

- Research the DELTA-EPSILON definition of a limit
- Tell me what it is and be able to use it
- MAX 2 page hand written report (plus graphs plus algebra) +2 Q quiz

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