Lesson 14 – Graphs of Rational Functions

Math HL - Santowski

Lesson Objectives

- The next class of functions that we will investigate is the rational functions. We will explore the following ideas:
- Definition of rational function.
- The basic (untransformed) rational function.
- Table of x and y values.
- Rational function and its inverse (algebraic and graphical examples).
- How does y change as (positive) x gets very large and very small?
- Graph of the basic rational function.
- Domain (and restrictions) and range of the basic rational function.
- Notion of a limit.

(A) Rational Functions

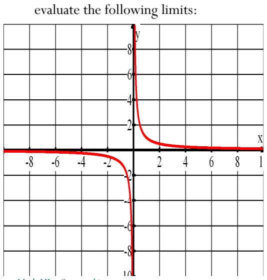
- Just as we saw how the Division Algorithm for integers applies to polynomials or functions, the definition of rational numbers can be extended to functions.
- A rational function is any function of the form r(x) = n(x)/d (x), where n(x) and d(x) represent numerator and denominator polynomials.
- Rational functions then are RATIOS of polynomials.

(B) The Basic Rational Function

- The statement $x \to \infty$ is read "x approaches infinity".
- To approach a value means to get close to the value but not necessarily equal to the value.
- What (single) value does y approach as $x \to \infty$?
- In calculus terms, this is said as "evaluate the limit as $x \to \infty$ of y".
- In calculus notation, this is written as: $\lim\limits_{x\to\infty} \frac{1}{x}$
- The statement $x \to 0$ is read "x approaches zero".
- From the table of positive x –values, what value does y approach as $x \to 0$?

(B) The Basic Rational Function

- Use the graph of f(x) =1/x to evaluate the following limits:



- (a) $\lim\limits_{x\to+\infty} \frac{1}{x}$

(b $\lim\limits_{x\to-\infty} \frac{1}{x}$

(a) $\lim\limits_{x\to0^-} \frac{1}{x}$

(a) $\lim\limits_{x\to0^+} \frac{1}{x}$

(a) $\lim\limits_{x\to2} \frac{1}{x}$

(a) $\lim\limits_{x\to-\frac{4}{3}} \frac{1}{x}$

(C) Transformational Form

- Transformational Method of Graphing Rational Functions
- To determine the transformations of any function, the functional equation must first be rewritten in transformational form,

$$f(x) = a\left(\frac{1}{b(x+c)}\right) + d$$

- Where a = vertical stretch/compression/reflection
- Where b = horizontal stretch/compression/reflection
- Where c = horizontal translation
- Where d = vertical translation

(C) Transformational Form

- To put rational functions into transformational form, we must recall the Division Algorithm, which states that
- y = n(x)/d(x) = Q(x) + r(x)/d(x)
- Example: Rewrite y =(5x + 2)/(x +1) in transformational form.
- Then state the transformations, apply them to the basic rational function y = 1/x and sketch y = (5x + 2)/(x +1).

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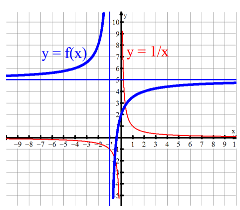
- y = n(x)/d(x) = Q(x) + r(x)/d(x)
- Example: Rewrite y =(5x + 2)/(x +1) in transformational form.
- Then state the transformations, apply them to the basic rational function y = 1/x and sketch y = (5x + 2)/(x +1).
- After division, $y = 5 + \frac{-3}{x+1} = -3\left(\frac{1}{x+1}\right) + 5$

(C) Transformational Form

- So our equation has become

$$y = 5 + \frac{-3}{x+1} = -3\left(\frac{1}{x+1}\right) + 5$$



- And our transformations of y = 1/x are:
- (a) reflected across the x-axis
- (b) a vertical stretch by a factor of 3
- (c) translated vertically up 5
- (d) translated left by 1

(D) Asymptotic Behaviour:

- Asymptotic Behaviour: The behaviour of the y – value of graphs to reach bounded values (limits) or to grow unboundedly (to approach infinity) is called "Asymptotic Behaviour".
- Asymptotes are boundary lines (or curves) that act as limiters or attractors of the shape of the graph.
- Some types of asymptotes may be crossed by the graph and while other types of asymptotes will never be crossed by the graph.
- Elements of graphs are pieces of information that enables us to sketch the graph.
- Asymptotes are examples of elements of rational functions.
- Other examples are intercepts and holes in the graph.

(D) Asymptotic Behaviour

- Vertical Asymptote: What value(s) of x make(s) y get very large?
- a) A vertical asymptote of a (rational) function is the vertical line x = h such that as x → h , y → ± ∞.
- b) There may be ONE or MORE THAN ONE VA (vertical asymptote) for rational functions.
- c) To determine the equation of the vertical asymptote(s), first ensure that you have checked for common factors and reduced the rational function. Then, set the denominator of a rational function equal to zero d(x) = 0 and solve for x. State VAs as equations of vertical lines.
- d) The graph of a rational function NEVER crosses a vertical asymptote.
- Exercise: Determine the equation(s) of the VA of y = x/(x^2 + x - 6).

(D) Asymptotic Behaviour

- Non-Vertical Asymptotes: What happens to y as x gets very large?
- a) Non-vertical asymptotes have equations of the form y =Q(x) , where Q(x) represents the quotient of the numerator divided by the denominator.
- b) There will be ONLY ONE non-vertical asymptote for rational functions.
- c) To determine the equation of the non-vertical asymptote, first ensure that you have checked for common factors and reduced the rational function. Then, set y equal to the quotient of the rational function's numerator divided by its denominator. Therefore y =Q(x) is the equation of the non-vertical asymptote AS LONG AS R(x) ≠ 0.
- d) The graph of a rational function MAY cross a non-vertical asymptote, but it does not have to.
- e) The order of the quotient (and therefore the shape of the non-vertical asymptote) depends on the orders of the numerator and denominator.
- Let's investigate three possible cases for the orders of n(x) and d(x) :

(D) Asymptotic Behaviour

- Case I/ Horizontal Asymptote: $R(x) = \frac{3x-6}{x+1}$
- y =Q(x) has the form y = k , where k εR .
- A horizontal asymptote of a (rational) function is the horizontal line y = k such that as x ⟶ ±∞, y ⟶k
- If there are no common factors and the order of n(x) is
  - • LESS THAN the order of d(x) , then there is a HA @ y = 0
  - • EQUAL TO the order of d(x) , then there is a HA @ y = k, k ≠ 0.

(D) Asymptotic Behaviour

- Case II/ Slant Asymptote: $R(x) = \frac{2x^2-x-3}{x+2}$
- y =Q(x) has the form y =mx +b, where m,bεR .
- The order of n(x) is ONE GREATER THAN the order of d(x) (as long as there are no common factors).
- A slant asymptote is the line y =mx +b such that as

  x ⟶ ±∞,y ⟶ mx +b

(D) Asymptotic Behaviour

- Case III/ Parabolic Asymptote: $R(x) = \frac{x^3+4x^2-4x-4}{x+3}$
- y =Q(x) has the form y = ax²+bx + c ; a,b,cεR .
- The order of n(x) is TWO GREATER THAN the order of d(x) (with no common factors).
- A parabolic asymptote of a (rational) function is the parabola y = ax² +bx + c, such that as x⟶±∞, y ⟶ ax² +bx + c .

(D) Asymptotic Behaviour

- Exercise: Determine the type and equation of the non-VA of:

$$y = \frac{2x}{x^2+x-6}$$
$$y = \frac{2x^2}{x^2+x-6}$$
$$y = \frac{2x^2}{x-6}$$

Preview Question

- Determine the value of:

$$\lim_{x\to 1}\left(\frac{x^2+3x-4}{x^2-x}\right)$$

(E) Holes in the Graph

- 1. Holes in the Graph: If the numerator and denominator have a common factor, then the rational function will have a hole in the graph. (There will be the same number of holes in the graph as there are common factors.)
- a) A hole in the graph is really the absence of a point on the graph at a particular set of coordinates.
- b) Fully factor every rational function to test for holes in the graph. If there are common factors, reduce the fraction (the rational function). Then, determine the root associated with the common factor and rewrite the rational function as follows:

$$r(x) = \frac{x^2-1}{x^2+3x+2} = \frac{(x+1)(x-1)}{(x+1)(x+2)} = \frac{x-1}{x+2}, x \neq -1,-2$$

(E) Holes in Graphs

- c) The x – coordinate of the hole is the root of the common factor: x_{hole} = -1 .
- d) The y–coordinate of the hole is result of the x – value substituted into the reduced rational function: y_{hole} = r(-1) = ((-1) – 1)/(-1 + 2) = -2/1 = -2.
- (e) Therefore the coordinates of the hole are (-1,-2) , which is to say that for the graph, (x,y) ≠ (-1, -2). To remove this point from the graph (or to "plot a hole"), sketch a small circle at the coordinates of the hole and sketch the rest of the graph as usual.
- Exercise: Determine the simplified form of the equation of the rational function y = (2x + 6)/ (x^2 - 4x - 21) and the coordinates of the hole.

(E) Holes in Graphs

- In calculus, we write the y –coordinate of the hole as $\lim_{x \to c} f(x)$ where x = c is the root of the factor common to the numerator and the denominator.
- Therefore, for this example, the y-coordinate of the hole is identified by the limit

$$\lim_{x \to -1} \frac{x^2 - 1}{x^2 + 3x + 2} = \lim_{x \to -1} \frac{(x-1)(x+1)}{(x+2)(x+1)} = \lim_{x \to -1} \frac{\text{x - 1}}{\text{x + 2}} = \frac{-1-1}{-1+2} = -2$$

(F) Exercises

- For each of the following, determine (where possible)
- a) the coordinates of any holes in the graph
- b) the coordinates of the x – and y – intercepts
- c) the equation(s) of the vertical asymptote(s)
- d) the type and equation of the non-vertical asymptote

(a) $y = \frac{1}{x}$ (b) $y = \frac{1}{x-3}$ (c) $y = -\frac{4}{x-3}$

(d) $y = \frac{x+2}{x^2-4}$ (e) $y = \frac{x-2}{x+1}$ (f) $y = \frac{3}{5-2x}$

(g) $y = \frac{2x+2}{x^2-3x-4}$ (h) $y = \frac{x+3}{x^2-3x-4}$ (i) $y = -\frac{1-x^2}{x}$

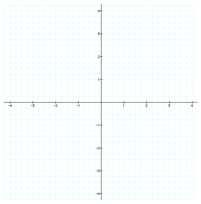
(j) $y = \frac{1}{x^2}$ (k) $y = \frac{6}{4-2x^2}$ (l) $y = \frac{x^2+1}{x^2-x-2}$

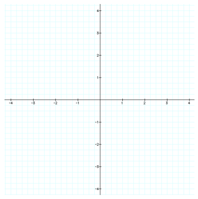
(G) Product of Signs

- Product of Signs Exercise: Perform the Product of Signs (Sign Chart) on (i) the quadratic function y = (x + 2)(x -1) and (ii) the rational function y = (x + 2)/(x – 1).
- a) Shade the regions where the graph does not lie.
- b) Are the shaded regions of the two functions the same or different? Explain.
- c) Are the boundary regions of the two functions the same or different? Explain.

(G) Product of Signs

- (i) the quadratic function y = (x + 2)(x -1)   (ii) the rational function y = (x + 2)/(x – 1).





(H) Elemental Method of Graphing Rational Functions (& the Product of Signs)

- Determine all of the elements of a rational function's graph by performing these steps in the following order.
- 1. Fully factor the numerator and denominator. If there are common factors, simplify the rational function and state the coordinates of the hole(s). Always work from the simplified rational function:
- 2. Determine the x – and y – intercepts.
- 3. Determine the equation(s) of the vertical asymptote(s): set d(x) = 0 and solve for x.
- 4. Determine the (one!) equation and type of the non-vertical asymptote: determine the quotient of n(x)/d(x) and set y =Q(x) .
- 5. Draw and xy – plane. Perform the Product of Signs on the (simplified) rational function and shade the regions where the graph WILL NOT LIE.
- 6. Plot all holes and intercepts. Sketch all asymptotes as dotted lines (the VA were already sketched when you did the boundary lines for the Product of Signs). Sketch the graph of the (simplified) rational function based on all of the elements that you have incorporated in the xy – plane.

(H) Elemental Method of Graphing Rational Functions (& the Product of Signs)

- Exercises: Sketch each of the following using the elemental method (see slide #31). ALSO sketch #2, 3 and 5 by the transformational method. For every graph …

- a) label any relevant points/holes
- b) draw all asymptotes as dotted lines with arrows on the ends
- c) state the domain and range

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(H) Elemental Method of Graphing Rational Functions (& the Product of Signs)

(a) $y = \frac{1}{x}$ (b) $y = \frac{1}{x-3}$ (c) $y = -\frac{4}{x-3}$

(d) $y = \frac{x+2}{x^2-4}$ (e) $y = \frac{x-2}{x+1}$ (f) $y = \frac{3}{5-2x}$

(g) $y = \frac{2x+2}{x^2-3x-4}$ (h) $y = \frac{x+3}{x^2-3x-4}$ (i) $y = -\frac{1-x^2}{x}$

(j) $y = \frac{1}{x^2}$ (k) $y = \frac{6}{4-2x^2}$ (l) $y = \frac{x^2+1}{x^2-x-2}$

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