Lesson 4 - Summation Notation & Infinite Geometric Series

Math HL1 - Santowski

(A) Review

- 1. Define a "sequence"
- 2. Define a "series"
- 3. State 2 formulas associated with arithmetic sequences and series
- 4. State 2 formulas associated with geometric sequences and series

(A) Review

- A sequence is a set of ordered terms, possibly related by some pattern, which could be defined by some kind of a "formula"
- One such pattern is called arithmetic because each pair of consecutive terms has a common difference
- A geometric sequence is one in which the consecutive terms differ by a common ratio

(B) REVIEW - Formulas

- For an arithmetic sequence, the formula for the general term is:

$$u_n = u_1 + (n-1)d$$

- For an arithmetic sequence then the formula for the sum of its terms is:

$$Sn = \frac{n}{2}\left(u_1 + u_n\right) = \frac{n}{2}\left(2u_1 + \left(n-1\right)d\right)$$

(B) REVIEW - Formulas

- For an geometric sequence then the formula for the general term is:

$$u_n = u_1 r^{n-1}$$

- So in general, the formula for the sum of a geometric series is:

$$S_n = \frac{(u_{n+1} - u_1)}{r-1} = \frac{u_1(r^n - 1)}{r-1}$$

(C) Summation Notation

- Summation notation is a shorthand way of saying take the sum of certain terms of a sequence ➔ the Greek letter sigma, Σ is used to indicate a summation
- In the expression $\sum_{i=1}^{n} a_i$
- *i* represents the index of summation (or term number), and a_i represents the general term of the sequence being summed
- So therefore, $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$

(D) Ex 1 - Summation Notation

- Ex 1 – List the terms of the series defined as below, then determine the sum

$$\sum_{i=1}^{6}(i+1)=\sum_{n=1}^{6}(n+1)$$

(D) Ex 1 - Summation Notation

- Ex 1 – Arithmetic Series

$$\sum_{n=1}^{6}(n+1)=(1+1)+(2+1)+(3+1)+(4+1)+(5+1)+(6+1)$$
$$\sum_{n=1}^{6}(n+1)=(2)+(3)+(4)+(5)+(6)+(7)$$
$$\sum_{n=1}^{6}(n+1)=27$$

(D) Ex 2 - Summation Notation

- Ex 1 – List the terms of the series defined as below, then determine the sum

$$\sum_{i=1}^{6}2(1.1)^{i}=\sum_{n=1}^{6}2(1.1)^{n}$$

(D) Ex 2 - Summation Notation

- Ex 1 – Geometric Series

$$\sum_{i=1}^{6}2(1.1)^{i}=\sum_{n=1}^{6}2(1.1)^{n}$$
$$\sum_{n=1}^{6}2(1.1)^{n}=2(1.1)^{1}+2(1.1)^{2}+2(1.1)^{3}+2(1.1)^{4}+2(1.1)^{5}+2(1.1)^{6}$$
$$\sum_{n=1}^{6}2(1.1)^{n}=2(1.1)+2(1.21)+2(1.331)+2(1.4641)+2(1.61051)+2(1.771561)$$
$$\sum_{n=1}^{6}2(1.1)^{n}=16.974342$$

(E) Examples of Summation Notation

- Write out the series expansion for the following and then evaluate the sums:

$$\sum_{i=1}^{8}i^{2}=$$
$$\sum_{i=1}^{10}\left(\frac{i-4}{i}\right)=$$
$$\sum_{i=4}^{9}e^{\ln(i)}=$$

(F) Word Problems with Geometric Sequences & Series - Annuities

- An annuity is simplistically a regular deposit made into an investment plan
- For example, I invest $2500 at the beginning of every year for ten years into an account that pays 9% p.a compounded annually at the end of the year.
- Determine the terms of the sequence.
- Determine the total value of my investment at the end of 10 years.

(F) Word Problems with Geometric Sequences & Series - Annuities

5. Find the total amount required to pay off a loan of $20,000 plus interest at the end of 5 years if the interest is compounded half yearly and the rate is 12%.

6. A man invests $500 at the beginning of each year in a superannuation fund. If the interest is paid at the rate of 12% p.a. on the investment (compounding annually), how much will his investment be worth after 20 years?

7. A woman invests $2000 at the beginning of each year into a superannuation fund for a period of 15 years at a rate of 9% p.a. (compounding annually). Find how much her investment is worth at the end of the 15 years.

8. A man deposits $3 000 annually to accumulate at 9% p.a. compound interest. How much will he have to his credit at the end of 25 years? Compare this to depositing $750 every three months for the same length of time and at the same rate. Which of these two options gives the better return?

(G) Infinite Geometric Series

- Show that the sum of *n terms of the series 2 + 1 + ½ + ¼ + ... u_n is always* less than 4, *where n is any natural number.*
- *So this will lead us to the idea of "formula" we could use to predict the sum of an infinite number of terms of a geometric series, provided that (or under the condition that ……)*

(G) Infinite Geometric Series

- *Explain what the following notations mean:*

$$S_6 = \sum_{i=1}^{6}\left(2^{2-i}\right) =$$

$$S_n = \sum_{i=1}^{n}\left(2^{2-i}\right) =$$

$$S_n = \lim_{n\to\infty}\sum_{i=1}^{n}\left(2^{2-i}\right) =$$

(F) Examples of Summation Notation

- Example – List several terms of the infinite series defined as below, then attempt to determine the sum.
- Now explain the what it means to say that a series CONVERGES or a series DIVERGES

- (a) $\sum_{n=1}^{\infty}\frac{1}{2^n} =$

  (b) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n} =$

  (c) $\sum_{n=1}^{\infty}(-2)^n =$

  (d) $\sum_{n=1}^{\infty}(2)^n =$

(H) Examples of Infinite Geometric Series

- Find the sum of the following infinite series, if possible. If not, explain why not.

(a) $\sum_{i=0}^{\infty}2\left(\frac{2}{3}\right)^i$ (b) $\sum_{i=0}^{\infty}4(0.75)^i$

(c) $\sum_{i=0}^{\infty}-10\left(\frac{1}{5}\right)^i$ (d) $\sum_{i=0}^{\infty}2\left(\frac{7}{3}\right)^i$

(G) Infinite Geometric Series

Evaluate:

(i) $27 + 9 + 3 + \frac{1}{3} + \ldots$ (ii) $1 - \frac{3}{10} + \frac{9}{100} - \frac{27}{1000} + \ldots$

(iii) $500 + 450 + 405 + 364.5 + \ldots$ (iv) $3 - 0.3 + 0.03 - 0.003 + 0.0003 - \ldots$

Combination of AS & GS

2. An arithmetic series has a first term of 2 and a fifth term of 30. A geometric series has a common ratio of –0.5. The sum of the first two terms of the geometric series is the same as the second term of the arithmetic series. What is the first term of the geometric series?

3. An arithmetic series has a first term of –4 and a common difference of 1. A geometric series has a first term of 8 and a common ratio of 0.5. After how many terms does the sum of the arithmetic series exceed the sum of the geometric series?

Combination of AS & GS

5. Bo-Youn and Ken are to begin a savings program. Bo-Youn saves $1 in the first week $2 in the second week, $4 in the third and so on, in geometric progression. Ken saves $10 in the first week, $15 in the second week, $20 in the third and so on, in arithmetic progression. After how many weeks will Bo-Youn have saved more than Ken?

6. Ari and Chai begin a training program. In the first week Chai will run 10km, in the second he will run 11km and in the third 12km, and so on, in arithmetic progression. Ari will run 5km in the first week and will increase his distance by 20% in each succeeding week.
(a) When does Ari's weekly distance first exceed Chai's?
(b) When does Ari's total distance first exceed Chai's?