Example 4 If $y = \tan^{-1}\left(\frac{x}{y}\right)$ find $\frac{dy}{dx}$.

Solution

$$\frac{dy}{dx} = \frac{1}{1 + \left(\frac{x}{y}\right)^2}\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\frac{dy}{dx} = \frac{1}{1 + \frac{x^2}{y^2}}\,\frac{y\frac{d}{dx}(x) - x\frac{d}{dx}(y)}{y^2}$$

$$\frac{dy}{dx} = \frac{y^2}{x^2 + y^2}\,\frac{y(1) - x\frac{dy}{dx}}{y^2}$$

$$(x^2 + y^2)\frac{dy}{dx} = y - x\frac{dy}{dx}$$

$$(x^2 + y^2 + x)\frac{dy}{dx} = y$$

$$\frac{dy}{dx} = \frac{y}{x^2 + y^2 + x}$$

EXERCISE 7.6

B 1. Find $\frac{dy}{dx}$ in each of the following.

(a) $y = \sin^{-1}(x + 1)$

(b) $y = \cos^{-1}(x^2)$

(c) $y = \tan^{-1}(3x)$

(d) $y = (\sin^{-1} x)^2$

(e) $y = \cos^{-1}\left(\frac{x^3}{2}\right)$

(f) $y = (1 + x^2)\tan^{-1} x$

(g) $y = \cos^{-1}\sqrt{2x - 1}$

(h) $y = \tan^{-1}(\sin x)$

(i) $y = \sin^{-1}\left(\frac{\cos x}{1 + \sin x}\right)$

(j) $y = \frac{\sin^{-1} x}{\cos^{-1} x}$

(k) $y = (\tan^{-1} x)^{-1}$

(l) $y = (\cos^{-1} x^2)^{-2}$

(m) $y = \tan^{-1} x + \tan^{-1}\frac{1}{x}$

(n) $y = \frac{\sqrt{1 - x^2}}{x} + \sin^{-1} x$

(o) $y = \frac{x}{\sqrt{1 - x^2}} - \sin^{-1} x$

(p) $y = \sin^{-1} x + \cos^{-1}\sqrt{1 - x^2}$

(q) $y = \sin(\sin^{-1} x^2)$

(r) $y = \sin^{-1}(\tan^{-1} x)$

(s) $y = x^2 \cos^{-1}\left(\frac{2}{x}\right)$

2. Find the slope of the tangent line to $f(x) = x \tan^{-1} x$ at the point where $x = 1$.
3. Find the equation of the tangent line to
$f(x) = x \sin^{-1}\left(\frac{x}{4}\right) + \sqrt{16 - x^2}$ at the point where $x = 2$.
4. If $f(x) = (3 \tan^{-1} x)^4$, find $f'(\sqrt{3})$.
5. If $y^2 \sin x = \tan^{-1}x - y$ find y'.
6. If $f(x) = (x - 3)\sqrt{6x - x^2} + 9 \sin^{-1}\left(\frac{x - 3}{3}\right)$ find $f'(3)$.
7. Differentiate and use your result to sketch the graph of the given function.
   (a) $y = \sin^{-1}(\cos x)$ (b) $y = \cos^{-1}(\cos x)$
   (c) $y = \sin^{-1}(\sin x)$ (d) $y = \sin^{-1}(\cos 2x)$

7.7 REVIEW EXERCISE

1. Evaluate each of the following limits.
   (a) $\lim_{x\to 0} \frac{\sin \frac{1}{2}x}{x}$ (b) $\lim_{x\to 0} \frac{\cos\left(\frac{\pi}{2} - x\right)}{x}$ (c) $\lim_{x\to 0} \frac{\sin 3x}{5x}$
   (d) $\lim_{x\to 0} \frac{\sin^2 x}{x \cos x}$ (e) $\lim_{x\to 0} x \csc x$ (f) $\lim_{x\to 0} \frac{\sin x \cos x}{x}$
   (g) $\lim_{x\to 0} \frac{1 - \cos 2x}{x}$ (h) $\lim_{x\to 0} \frac{2 \tan^2 x}{x^2}$ (i) $\lim_{x\to \frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{\cos x}$
2. Differentiate y with respect to x.
   (a) $y = \tan^4 3x$ (b) $y = \frac{\sin x}{1 - 2 \cos x}$ (c) $y = \sec x^2$
   (d) $y = \frac{\cot^2 2x}{1 + x^2}$ (e) $y = \csc(x^3 + 1)$ (f) $y = 2 \sec \sqrt{x}$
   (g) $y = \sqrt[3]{x} \tan x$ (h) $y = \cos^2(\tan x)$
   (i) $y = \frac{1}{\sin(x - \sin x)}$
3. Find $\frac{dy}{dx}$ by implicit differentiation.
   (a) $y = \cos(x - y)$ (b) $\sin(x + y) + \sin(x - y) = 1$
   (c) $y = \tan(x + y)$ (d) $\cos(x + y) = y \sin x$
   (e) $\cot xy + xy = 0$ (f) $\csc(x - y) + \sec(x + y) = x$
4. At what points on the curve $y = \sin x + \cos x$, $0 \leq x \leq 2\pi$, is the tangent line horizontal?