12.7 STRATEGY FOR TESTING SERIES

We now have several ways of testing a series for convergence or divergence; the problem is to decide which test to use on which series. In this respect, testing series is similar to integrating functions. Again there are no hard and fast rules about which test to apply to a given series, but you may find the following advice of some use.

It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its *form*.

1. If the series is of the form $\Sigma\, 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.
2. If the series has the form $\Sigma\, ar^{n-1}$ or $\Sigma\, ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. Notice that most of the series in Exercises 12.4 have this form. (The value of p should be chosen as in Section 12.4 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\Sigma\, a_n$ has some negative terms, then we can apply the Comparison Test to $\Sigma\, |a_n|$ and test for absolute convergence.
4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.
5. If the series is of the form $\Sigma\, (-1)^{n-1}b_n$ or $\Sigma\, (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.
6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.
7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.
8. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

In the following examples we don't work out all the details but simply indicate which tests should be used.

EXAMPLE 1 $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

Since $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$, we should use the Test for Divergence. □

EXAMPLE 2 $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

Since a_n is an algebraic function of n, we compare the given series with a p-series. The comparison series for the Limit Comparison Test is $\Sigma\, b_n$, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$

EXAMPLE 3 $\sum_{n=1}^{\infty} ne^{-n^2}$

Since the integral $\int_1^{\infty} xe^{-x^2}\,dx$ is easily evaluated, we use the Integral Test. The Ratio Test also works.

EXAMPLE 4 $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

Since the series is alternating, we use the Alternating Series Test.

EXAMPLE 5 $\sum_{k=1}^{\infty} \frac{2^k}{k!}$

Since the series involves $k!$, we use the Ratio Test.

EXAMPLE 6 $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

Since the series is closely related to the geometric series $\Sigma\, 1/3^n$, we use the Comparison Test.

12.7 EXERCISES

1–38 Test the series for convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{1}{n+3^n}$

2. $\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$

3. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

4. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$

5. $\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$

6. $\sum_{n=1}^{\infty} \frac{1}{2n+1}$

7. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

8. $\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$

9. $\sum_{k=1}^{\infty} k^2 e^{-k}$

10. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

11. $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$

12. $\sum_{n=1}^{\infty} \sin n$

13. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

14. $\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$

15. $\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}$

16. $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

17. $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$

18. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

19. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$

20. $\sum_{k=1}^{\infty} \frac{k+5}{5^k}$

21. $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$

22. $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

23. $\sum_{n=1}^{\infty} \tan(1/n)$

24. $\sum_{n=1}^{\infty} n \sin(1/n)$

25. $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

26. $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

27. $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$

28. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

29. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh n}$

30. $\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$

31. $\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$

32. $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

33. $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$

34. $\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$

35. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

36. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

37. $\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)^n$

38. $\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)$

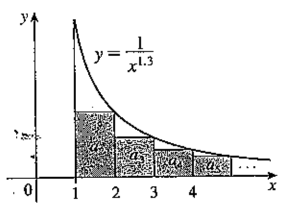
71. $\{s_n\}$ is bounded and increasing.

73. (a) $0, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}, 1$

75. (a) $\frac{1}{2}, \frac{5}{6}, \frac{23}{24}, \frac{119}{120}; \dfrac{(n+1)! - 1}{(n+1)!}$ (c) 1

EXERCISES 12.3 ■ PAGE 739

1. C



3. C **5.** C **7.** C **9.** D **11.** C **13.** D **15.** C **17.** C **19.** C **21.** D **23.** C **25.** C **27.** $p > 1$ **29.** $p < -1$ **31.** $(1, \infty)$

33. (a) 1.54977, error $\leqslant 0.1$ (b) 1.64522, error $\leqslant 0.005$ (c) $n > 1000$

35. 0.00145 **41.** $b < 1/e$

EXERCISES 12.4 ■ PAGE 745

1. (a) Nothing (b) C **3.** C **5.** D **7.** C **9.** C **11.** C **13.** C **15.** C **17.** D **19.** D **21.** C **23.** C **25.** D **27.** C **29.** C **31.** D

33. 1.249, error < 0.1 **35.** 0.76352, error < 0.001

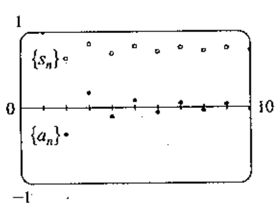
45. Yes

EXERCISES 12.5 ■ PAGE 749

1. (a) A series whose terms are alternately positive and negative (b) $0 < b_{n+1} \leqslant b_n$ and $\lim_{n\to\infty} b_n = 0$, where $b_n = |a_n|$ (c) $|R_n| \leqslant b_{n+1}$

3. C **5.** C **7.** D **9.** C **11.** C **13.** D **15.** C **17.** C **19.** D

21. 1.0000, 0.6464, 0.8389, 0.7139, 0.8033, 0.7353, 0.7893, 0.7451, 0.7821, 0.7505; error < 0.0275



23. 5 **25.** 4 **27.** 0.9721 **29.** 0.0676

31. An underestimate **33.** p is not a negative integer

35. $\{b_n\}$ is not decreasing

EXERCISES 12.6 ■ PAGE 755

Abbreviations: AC, absolutely convergent; CC, conditionally convergent

1. (a) D (b) C (c) May converge or diverge

3. AC **5.** CC **7.** AC **9.** D **11.** AC **13.** AC **15.** AC **17.** CC **19.** AC **21.** AC **23.** D **25.** AC **27.** D **29.** D **31.** (a) and (d)

35. (a) $\frac{661}{960} \approx 0.68854$, error < 0.00521
(b) $n \geqslant 11$, 0.693109

EXERCISES 12.7 ■ PAGE 758

1. C **3.** D **5.** C **7.** D **9.** C **11.** C **13.** C **15.** C **17.** D **19.** C **21.** C **23.** D **25.** C **27.** C **29.** C **31.** D **33.** C **35.** C **37.** C

EXERCISES 12.8 ■ PAGE 763

1. A series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$, where x is a variable and a and the c_n's are constants

3. $1, [-1, 1)$ **5.** $1, [-1, 1]$ **7.** $\infty, (-\infty, \infty)$

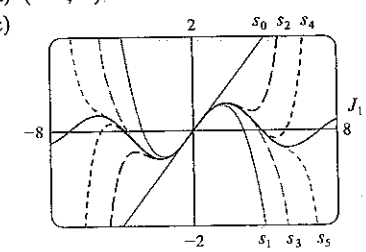
9. $\frac{1}{4}, \left(-\frac{1}{4}, \frac{1}{4}\right)$ **11.** $\frac{1}{2}, \left(-\frac{1}{2}, \frac{1}{2}\right]$ **13.** $4, (-4, 4]$

15. $1, [1, 3]$ **17.** $\frac{1}{3}, \left[-\frac{13}{3}, -\frac{11}{3}\right)$ **19.** $\infty, (-\infty, \infty)$

21. $b, (a-b, a+b)$ **23.** $0, \left\{\frac{1}{2}\right\}$ **25.** $\frac{1}{4}, \left[-\frac{1}{2}, 0\right]$

27. $\infty, (-\infty, \infty)$ **29.** (a) Yes (b) No **31.** k^k **33.** No

35. (a) $(-\infty, \infty)$
(b), (c)



37. $(-1, 1), f(x) = (1 + 2x)/(1 - x^2)$ **41.** 2

EXERCISES 12.9 ■ PAGE 769

1. 10 **3.** $\sum_{n=0}^{\infty} (-1)^n x^n, (-1, 1)$ **5.** $2\sum_{n=0}^{\infty} \frac{1}{3^{n+1}} x^n, (-3, 3)$

7. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{9^{n+1}} x^{2n+1}, (-3, 3)$ **9.** $1 + 2\sum_{n=1}^{\infty} x^n, (-1, 1)$

11. $\sum_{n=0}^{\infty} \left[(-1)^{n+1} - \frac{1}{2^{n+1}}\right] x^n, (-1, 1)$

13. (a) $\sum_{n=0}^{\infty} (-1)^n (n+1) x^n, R = 1$

(b) $\frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n, R = 1$

(c) $\frac{1}{2} \sum_{n=2}^{\infty} (-1)^n n(n-1) x^n, R = 1$

15. $\ln 5 - \sum_{n=1}^{\infty} \frac{x^n}{n5^n}, R = 5$ **17.** $\sum_{n=3}^{\infty} \frac{n-2}{2^{n-1}} x^n, R = 2$