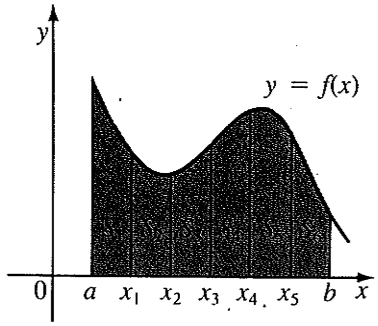
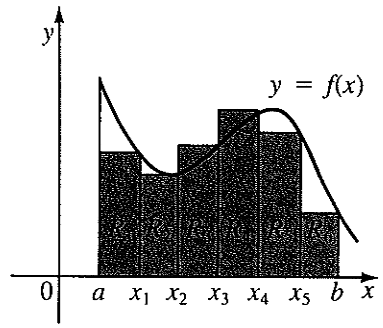
10.4 AREAS AS LIMITS

In this section we examine a different method of calculating area. The region whose area we wish to calculate is divided into narrow strips and a rectangle is used to approximate the area of each strip. The sum of the areas of these rectangles approximates the area of the region.





The region under $y = f(x)$ from a to b has been divided into six strips, $S_1, S_2, \ldots, S_6$. Rectangles, $R_1, R_2, \ldots, R_6$, have been constructed to approximate the area of each strip. In this case, the base of the rectangle is the width of the strip and the height of the rectangle is the value of the function at the right-hand endpoint of each strip. Rectangle R_3 has width $x_3 - x_2$ and height $f(x_3)$.

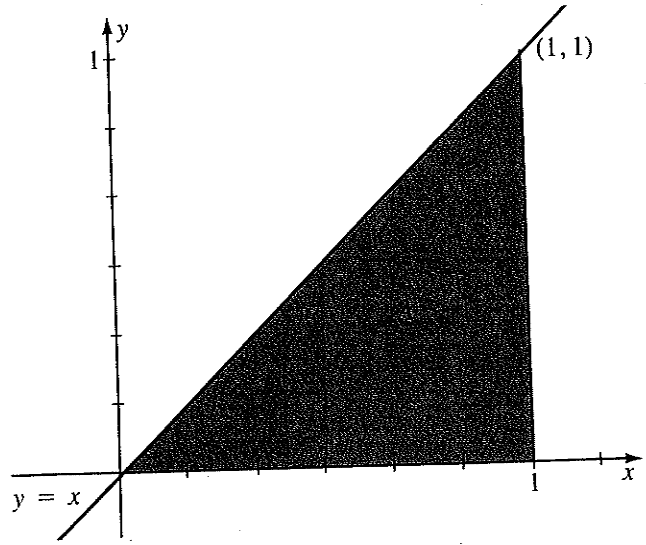
The left-hand endpoint or any point in between, such as the midpoint of the base, could have been chosen to determine the height of a rectangle but we will be using the right-hand endpoint exclusively.

Example 1 (a) Calculate the area under $y = x$, from 0 to 1.

(b) Approximate the same area by subdividing the region into six strips of equal width and finding the sum of the areas of the rectangles determined by the right-hand endpoint of each interval.

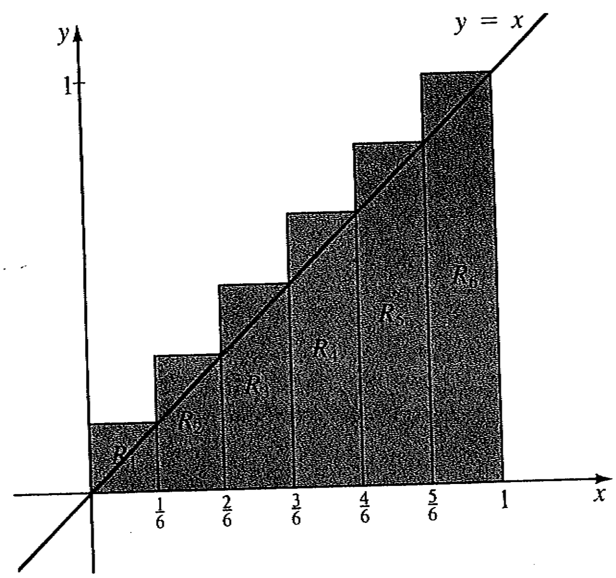
(c) Repeat part (b) using twelve strips of equal width.

Solution (a)



The required area is a right triangle with base 1 and height 1. Since the formula for the area of a triangle is $\frac{1}{2}bh$, the required area is $\frac{1}{2}(1)(1) = \frac{1}{2} = 0.5$.

(b)



The required area is approximately the sum of the areas of the rectangles R_1, R_2, R_3, R_4, R_5, and R_6.

Now the width of each rectangle is $\frac{1}{6}$.

So, Area $\doteq \frac{1}{6}f\left(\frac{1}{6}\right) + \frac{1}{6}f\left(\frac{2}{6}\right) + \frac{1}{6}f\left(\frac{3}{6}\right) + \frac{1}{6}f\left(\frac{4}{6}\right) + \frac{1}{6}f\left(\frac{5}{6}\right) + \frac{1}{6}f\left(\frac{6}{6}\right)$

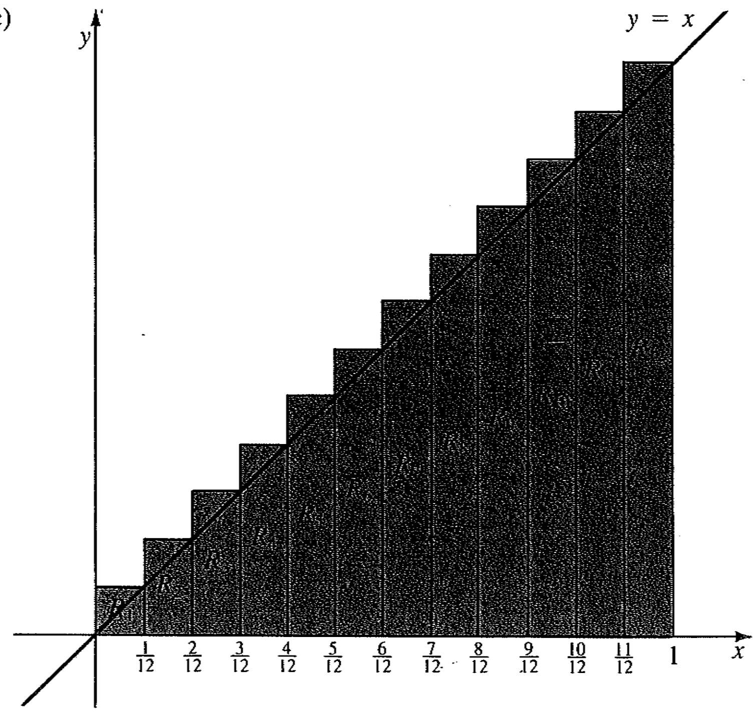
$$= \frac{1}{6}\left(\frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}\right)$$

$$= \frac{1}{36}(1 + 2 + 3 + 4 + 5 + 6)$$

$$= \frac{21}{36}$$

$$\doteq 0.583$$

(c)



The required area is approximately the sum of the areas of the rectangles $R_1, R_2, R_3, \ldots, R_{12}$.

Now the width of each rectangle is $\frac{1}{12}$.

So, Area $\doteq \frac{1}{12}\left[f\left(\frac{1}{12}\right) + f\left(\frac{2}{12}\right) + f\left(\frac{3}{12}\right) + \ldots + f\left(\frac{12}{12}\right)\right]$

$$= \frac{1}{12}\left(\frac{1}{12} + \frac{2}{12} + \frac{3}{12} + \ldots + \frac{12}{12}\right)$$

$$= \frac{1}{144}(1 + 2 + 3 + \ldots + 12)$$

$$= \frac{1}{144}\left(\frac{12 \times 13}{2}\right)$$

$$= \frac{78}{144}$$

$$\doteq 0.542$$

Intuitively, the more rectangles that are constructed the better the approximation becomes. Suppose we constructed 100 rectangles of equal width $\frac{1}{100}$.

$$
\begin{aligned}
\text{Area} &\doteq \frac{1}{100}\left[f\left(\tfrac{1}{100}\right) + f\left(\tfrac{2}{100}\right) + f\left(\tfrac{3}{100}\right) + \ldots + f\left(\tfrac{100}{100}\right)\right] \\
&\doteq \tfrac{1}{100}\left(\tfrac{1}{100} + \tfrac{2}{100} + \tfrac{3}{100} + \ldots + \tfrac{100}{100}\right) \\
&= \tfrac{1}{100^2}(1 + 2 + 3 + \ldots + 100) \\
&= \tfrac{1}{100^2}\left(\tfrac{100 \times 101}{2}\right) \\
&= \tfrac{101}{200} \\
&= 0.505
\end{aligned}
$$

This is much closer to the actual area of 0.5 calculated in Example 1(a).

If we subdivide the region into n strips of equal width $\frac{1}{n}$ we could construct n rectangles and

$$
\begin{aligned}
\text{Area} &\doteq \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \ldots + f\left(\frac{n}{n}\right)\right] \\
&= \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \ldots + \frac{n}{n}\right) \\
&= \frac{1}{n^2}(1 + 2 + 3 + \ldots + n) \\
&= \frac{1}{n^2}\left[\frac{n(n + 1)}{2}\right] \\
&= \frac{n^2 + n}{2n^2}
\end{aligned}
$$

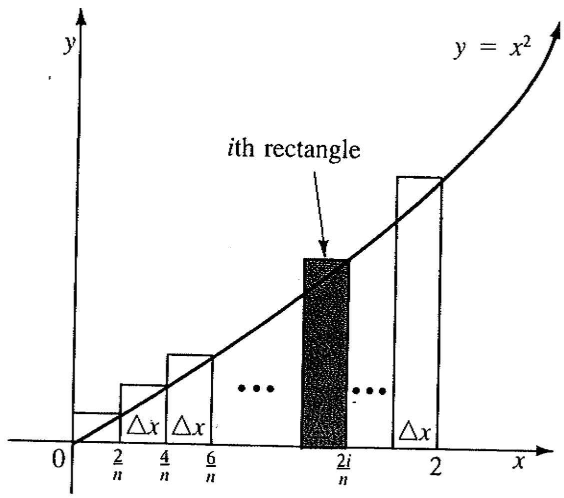
Now as $n \to \infty$ the number of rectangles increases and their width $\frac{1}{n}$ approaches 0 and the limit of the sum of the rectangles as $n \to \infty$ produces the actual area.

$$
\text{Area} = \lim_{n \to \infty} \frac{n^2 + n}{2n^2} = \lim_{n \to \infty} \frac{1 + \frac{1}{n}}{2} = \tfrac{1}{2} = 0.5
$$

We use sigma notation to write the series in the rest of the examples.

Example 2 Find the area under $y = x^2$ from $x = 0$ to $x = 2$.

Solution Subdivide the region into n strips of equal width, $\frac{2}{n}$. Consider a general rectangle, which we call the ith rectangle, having width $\Delta x = \frac{2}{n}$ and height $f\left(\frac{2i}{n}\right)$ determined by the right-hand endpoint.



Using sigma notation, the sum of the areas of the n rectangles is

$$\begin{aligned}
\sum_{i=1}^{n} \frac{2}{n} f\left(\frac{2i}{n}\right) &= \sum_{i=1}^{n} \left(\frac{2}{n}\right)\left(\frac{4i^2}{n^2}\right) \\
&= \frac{8}{n^3} \sum_{i=1}^{n} i^2 \\
&= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\
&= \frac{4(2n^2+3n+1)}{3n^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\text{Area} &= \frac{4}{3} \lim_{n \to \infty} \frac{2n^2+3n+1}{n^2} \\
&= \frac{4}{3} \lim_{n \to \infty} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \\
&= \tfrac{4}{3}(2+0+0) \\
&= \frac{8}{3}
\end{aligned}$$

Example 3 Find the area under $y = x^3 + x$ from $x = 1$ to $x = 4$.

Solution Subdivide the region into n strips of equal width

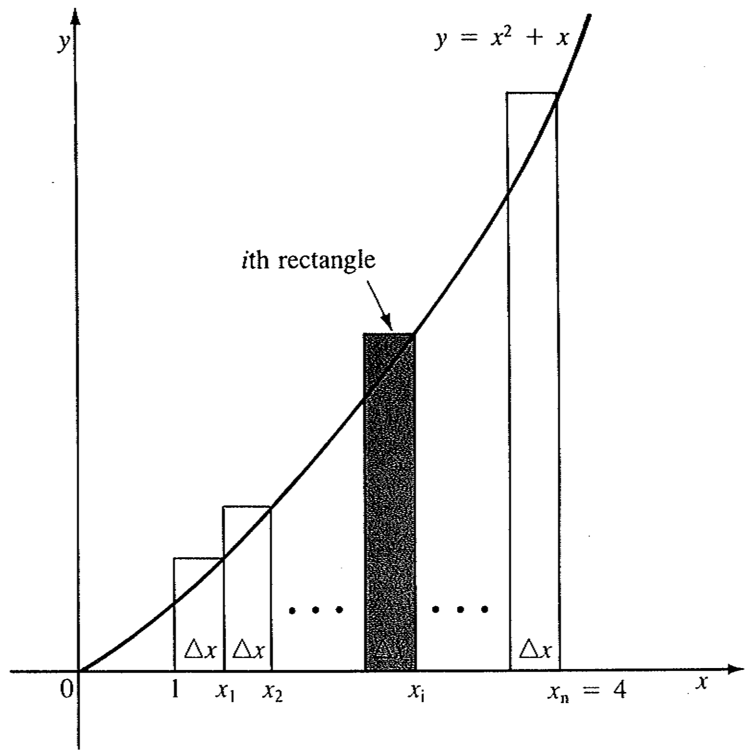
$$\Delta x = \frac{4-1}{n} = \frac{3}{n}$$

$$x_1 = 1 + \frac{3}{n}$$
$$x_2 = 1 + \frac{6}{n}$$
$$x_3 = 1 + \frac{9}{n}$$
$$\vdots$$

The ith rectangle has right-hand endpoint

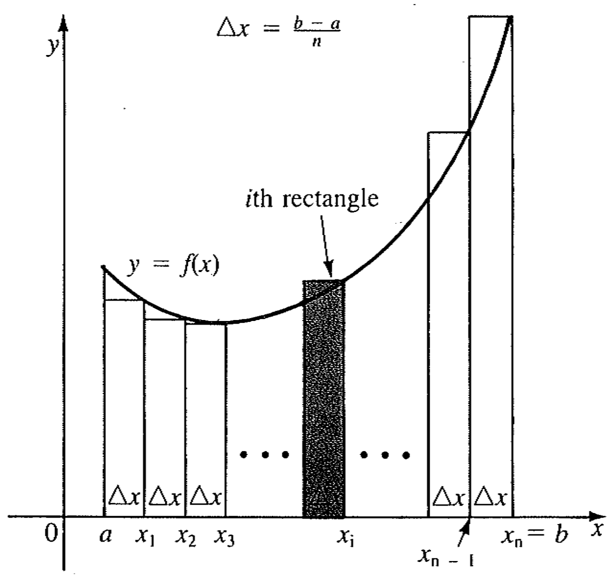
$$x_i = 1 + \frac{3i}{n}$$

and height $f\left(1 + \frac{3i}{n}\right)$



$$\begin{aligned}
\text{Area} &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} f\left(1 + \frac{3i}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(1 + \frac{3i}{n}\right)^3 + \left(1 + \frac{3i}{n}\right)\right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3}{n} + \frac{27i}{n^2} + \frac{81i^2}{n^3} + \frac{81i^3}{n^4} + \frac{3}{n} + \frac{9i}{n^2}\right) \\
&= \lim_{n\to\infty} \left(\frac{81}{n^4}\sum_{i=1}^{n} i^3 + \frac{81}{n^3}\sum_{i=1}^{n} i^2 + \frac{36}{n^2}\sum_{i=1}^{n} i + \frac{6}{n}\sum_{i=1}^{n} 1\right) \\
&= \lim_{n\to\infty} \left[\frac{81n^2(n+1)^2}{4n^4} + \frac{81n(n+1)(2n+1)}{6n^3} + \frac{36n(n+1)}{2n^2} + \frac{6n}{n}\right] \\
&= \lim_{n\to\infty} \left[\frac{81}{4}\left(1 + \frac{1}{n}\right)^2 + \frac{81}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 18\left(1 + \frac{1}{n}\right) + 6\right] \\
&= \frac{81}{4} + \frac{81}{3} + 18 + 6 \\
&= \frac{285}{4}
\end{aligned}$$

We develop a formula to find the area under $y = f(x)$ from a to b for f continuous and positive. We subdivide the region into n strips of equal width



From the diagram we see that the right-hand endpoints of the intervals are

$$x_1 = a + \Delta x$$
$$x_2 = a + 2\Delta x$$
$$x_3 = a + 3\Delta x$$
$$\vdots$$

The right-hand endpoint of the ith interval is

$$x_i = a + i\Delta x$$

The height of the ith rectangle is $f(x_i)$, so its area is

$$\text{height} \times \text{width} = f(x_i)\Delta x$$

To find the required area, we take the limit of the sums of the areas of the rectangles.

$$\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\text{where } \Delta x = \frac{b - a}{n} \text{ and } x_i = a + i\Delta x$$

EXERCISE 10.4

B 1. R is the region under $y = x + 1$ from 0 to 6.
   (a) Calculate the area of R using the formula for the area of a trapezoid.
   (b) Approximate the area of R by dividing it into six subintervals of equal width and summing the areas of rectangles.

2. R is the region under $y = x^2 + 1$ from 1 to 3.
   (a) Calculate the area of R using the differential equation $A'(x) = f(x)$.
   (b) Approximate the area of R by dividing it into ten subintervals of equal width and summing the areas of rectangles.

3. Use methods of this section to calculate the area of the given region.
   (a) under $y = x^3$ from 0 to 4
   (b) under $y = 2 + x^2$ from 0 to 3
   (c) under $y = x + 2x^3$ from 0 to 2
   (d) under $y = 3x^3 + 2x^2 + x$ from 0 to 1

10.5

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4. Use methods of this section to calculate the area of the given region.
(a) $y = -x^2 + 16$ from 1 to 3
(b) $y = x^2 + 3x - 2$ from 1 to 4
(c) $y = \frac{1}{2}x^3$ from 2 to 4
(d) $y = x^2 + x + 1$ from -1 to 3

5. Approximate the area under $y = \sin x$ from 0 to π by summing the areas of six rectangles of equal width.

C **6.** Find the area between the given curves by summing the areas of n rectangles of equal width.
(a) $y = x^2 + 4$ and $y = x + 2$ from $x = 0$ to $x = 2$
(b) $y = x^3 - 4x$ and $y = 5x$

PROBLEMS PLUS

(a) Show that

$$2 \sin \tfrac{1}{2}x \cos ix = \sin\left(i + \tfrac{1}{2}\right)x - \sin\left(i - \tfrac{1}{2}\right)x$$

(b) Use the identity in part (a) to show that

$$\sum_{i=1}^{n} \cos ix = \frac{\sin\left(n + \frac{1}{2}\right)x - \sin \frac{1}{2}x}{2 \sin \frac{1}{2}x}$$

(c) Deduce from part (b) that

$$\sum_{i=1}^{n} \cos ix = \frac{\sin \frac{1}{2}nx \cos \frac{1}{2}(n + 1)x}{\sin \frac{1}{2}x}$$

(d) Use part (c) to find the area under the curve $y = \cos x$ from 0 to b, $0 \leqslant b \leqslant \dfrac{\pi}{2}$, as a limit of sums.

10.5 NUMERICAL METHODS

We can accurately evaluate an area under $y = f(x)$ if we can find an antiderivative of f. Sometimes it is difficult or even impossible to find such an antiderivative. In such cases we can only approximate the area under the curve. In Section 10.4, we approximated the area of a region by subdividing it into narrow strips and approximating the area of each strip with a rectangle. The sum of the areas of the rectangles approximated the area of the region. As the widths of the strips became narrower, the number of rectangles increased and the approximation became better.