IM3L6.2 – Introduction to Base e | Unit 6 – Exp & Log Functions

(A) Lesson Context

BIG PICTURE of this UNIT:	• How do algebraically & graphically work with growth and decay applications? • What are logarithms and how do we invert or undo an exponential function? • How do we work with simple algebraic and graphic situations involving the use of logarithms (or inversing exponentials?)		
CONTEXT of this LESSON:	Where we've been We have introduced the natural base e & then developed the exponential function $f(t) = Pe^{rt}$	Where we are Consolidate & extend our ability to work with exponential eqns $f(n) = CB^n$ & $f(t) = Pe^{rt}$	Where we are heading How do work with the mathematically model $f(x) = AB^{k(x+c)} + d$?

(B) Lesson Objectives:

a. Consolidate & extend our ability to work with exponential eqns $f(n) = CB^n$ & $f(t) = Pe^{rt}$

(C) Graphing with Base e (Time Permitting)

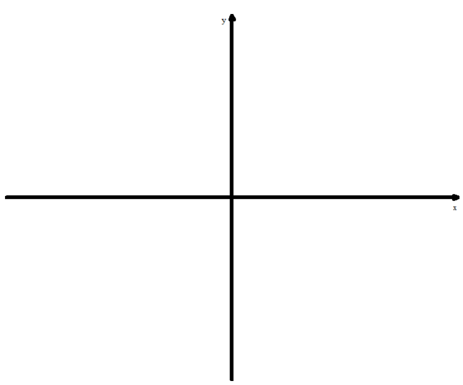
Graph Set A

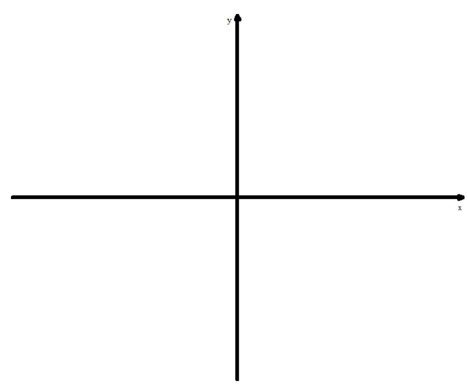
$y = e^x$

$y = e^{-x}$

$y = -e^x$

$y = -e^{-x}$





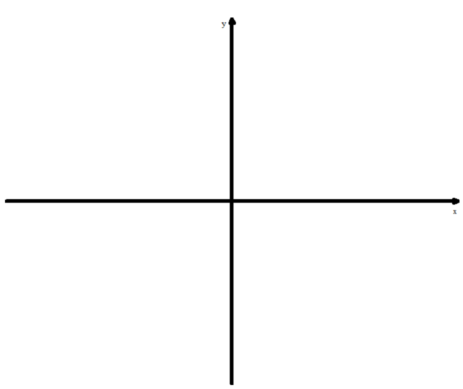
Graph Set B

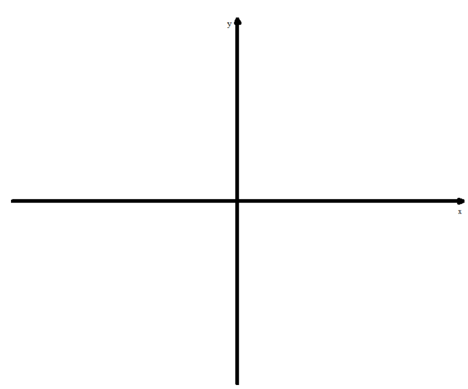
$y = e^x + 5$

$y = e^x - 5$

$y = 5 - e^x$

$y = 5 - e^{-x}$





Station # 1 – Exponential Functions and Compound Interest

Example # 1: The population of a town is modeled by the following exponential function. $P(t) = 15{,}752(1.045)^t$ where t is time in years since 1990. Please answer the quesitons below using the function given. (a) Is the population of the town growing or shrinking… by what percentage? (b) Find P(5). Interpret. (c) What was the population of the town in 1990? (d) When will the population of the town become more that 50,000? (e) What assumptions are we making in questions (d)?	**Example # 2:** You invest 6,500 Euro in a bank that gives you a 5.5% annual interest rate, compounded weekly. (a) Create an equation to model the growth of your principle investment. (b) What is your investment worth after 7 years? (c) How long will it take to triple your investment? (d) If you keep the same interest rate, and want to have 30,000 Euro after 20 years… how much should your Principle Investment be? Round to the nearest dollar.
Example # 3: There has been an accident at the local Nuclear Plant and a new radioactive material (Santoagen) has been spilled. This radioactive material begins to decay exponentially. There were 1820 (gens) of Santoagen which is way too radioactive to work with. 8 hours later there were 576 (gens) (a) What is the decay rate of Santoagen? (b) This material becomes non-leathal when there is a max of 20 (gens). When will it be safe for workers to enter the space and clear it out? (c) Will there ever be 0 (gens) left of the Santoagen material?	**Example # 4:**

Station # 2 – Working with Half Life and Doubling

Example #1: The population of a certain bacteria grows exponentially and can be modeled by the given equations P(t) where t is time in hours.

$$P(t) = 18(2)^{\frac{t}{3.5}}$$

Please answer the following questions give the function P(t)

(a) What was the population of the Bacteria when the observations started?

(b) What is the percentage Growth of this population? What do we also call this kind of growth?

(c) What is the double period of this bacteria? Explain what this means.

(d) How much bacteria will be present in 35 hours?

(e) When will the bacteria reach a population of 294,900?

Example # 3: The half life of caffeine in a child's system when a child eat or drinks something with cafine is 1.5 hours. How much caffine would remain in a childs body if the child ate a chocolate bar with 30 mg of caffine 8 hours before?

Example #2: The population of a bacterial culture doubles every 2.5 hours. An experiment begins with 512 bacteria. Determine the number of bacteria after

(a) 3 Hours

(b) 6 Hours

(c) 1 Day

(d) 1 Week

(e) If the growth rates remained constant, how long ago were there only 2 bacteria present in the experiment.

Example # 4: The half life of caffeine in a child's system when a child eat or drinks something with cafine is 1.5 hours. How much caffine would remain in a childs body if the child ate a chocolate bar with 30 mg of caffine 8 hours before?

Example # 5: The half-life of Carbon-14 is about 5370 years. What percentage of the original carbon-14 would you expect to find in a sample after 2500 years?

Station # 3 – Working with the Natural Base & Applications

Example #1: The number of bacteria in a culture is given by the function $n(t) = 10e^{0.22t}$

(a) What is the rate of growth of this bacterium population? Express your answer as a percentage.

(b) What is the initial population of the culture (at t = 0)?

(c) Evaluate and interpret n(15).

(d) Solve and interpret 500 = n(t).

(e) What is the doubling time for this bacterial population?

Example #2

60. Population Growth. If the population in Mexico is around 100 million people now and if the population grows continuously at an annual rate of 2.3%, what will the population be in 8 years? Compute the answer to two significant digits.

Example #3

65. Marine Biology. Marine life is dependent upon the microscopic plant life that exists in the *photic zone,* a zone that goes to a depth where about 1% of the surface light still remains. Light intensity I relative to depth d, in feet, for one of the clearest bodies of water in the world, the Sargasso Sea in the West Indies, can be approximated by

$$I = I_0 e^{-0.00942d}$$

where I_0 is the intensity of light at the surface. What percentage of the surface light will reach a depth of

(A) 50 feet? (B) 100 feet?

Example #4

62. Population Growth. In 1996 the population of Germany was 84 million and the population of Egypt was 64 million. If the populations of Germany and Egypt grow continuously at annual rates of −0.15% and 1.9%, respectively, when will Egypt have a greater population than Germany?

STATION #4 – Working with the Natural Base & Applications

Example #1:

67. Money Growth. If you invest \$5,250 in an account paying 11.38% compounded continuously, how much money will be in the account at the end of

(A) 6.25 years? (B) 17 years?

Example #2

71. Present Value. A promissory note will pay \$30,000 at maturity 10 years from now. How much should you be willing to pay for the note now if the note gains value at a rate of 9% compounded continuously?

Example #3

Advertising. A company is trying to expose a new product to as many people as possible through television advertising in a large metropolitan area with 2 million potential viewers. A model for the number of people N, in millions, who are aware of the product after t days of advertising was found to be

$$N = 2(1 - e^{-0.037t})$$

(A) How many viewers are aware of the product after 2 days? After 10 days? Express answers as integers, rounded to three significant digits.

(B) How many days will it take until half of the potential viewers will become aware of the product? Round answer to the nearest integer.

Example #4

Three years ago, the fish population in Loon Lake was 2500. Due to the effects of acid rain, there are now about 1950 fish in the lake. Assume that the decline of the fish population is exponential and is **continuous**. Find the predicted fish population 5 years from now.

Example #5: The population of a town is **continuously changing** and it appears to be increasing exponentially. Town planners need a model for predicting the future population. In the year 2000, the population was 35,000, while in the year 2010, the population grew to 57,010.

(a) Create an **<u>exponential</u>** algebraic model for the town's population growth.
(b) Check your population model by using the fact that the town's population was 72,825 in 2015.
(c) CALCULATE: What will be the town's population in 2030?

CHALLENGE QUESTIONS

15. Graph $y = \frac{e^x + e^{-x}}{2}$. (See the hanging rope on page 187.)

16. Sketch the graph of $y = e^{-x^2}$. (See the "bell-shaped curve" on page 187.)

C **19.** It can be proved that $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$.
Approximate e by using the first five terms. (*Note*: $n!$, read "n factorial," denotes $n(n-1)(n-2)\cdots 2 \cdot 1$. For example, $3! = 3 \cdot 2 \cdot 1 = 6$.)

C *It is common practice in many applications of mathematics to approximate nonpolynomial functions with appropriately selected polynomials. For example, the polynomials in Problems 53–56, called* ***Taylor polynomials,*** *can be used to approximate the exponential function $f(x) = e^x$. To illustrate this approximation graphically, in each problem graph $f(x) = e^x$ and the indicated polynomial in the same viewing window, $-4 \le x \le 4$ and $-5 \le y \le 50$.*

53. $P_1(x) = 1 + x + \frac{1}{2}x^2$

54. $P_2(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

55. $P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

56. $P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$

58. Investigate the behavior of the functions $g_1(x) = xe^x$, $g_2(x) = x^2e^x$, and $g_3(x) = x^3e^x$ as $x \to \infty$ and as $x \to -\infty$, and find any horizontal asymptotes. Generalize to functions of the form $g_n(x) = x^ne^x$, where n is any positive integer.

57. Investigate the behavior of the functions $f_1(x) = x/e^x$, $f_2(x) = x^2/e^x$, and $f_3(x) = x^3/e^x$ as $x \to \infty$ and as $x \to -\infty$, and find any horizontal asymptotes. Generalize to functions of the form $f_n(x) = x^n/e^x$, where n is any positive integer.

82. Training. A trainee is hired by a computer manufacturing company to learn to test a particular model of a personal computer after it comes off the assembly line. The learning curve for an average trainee is given by

$$N = \frac{200}{4 + 21e^{-0.1t}}$$

(A) How many computers can an average trainee be expected to test after 3 days of training? After 6 days? Round answers to the nearest integer.

(B) How many days will it take until an average trainee can test 30 computers per day? Round answer to the nearest integer.

(C) Does N approach a limiting value as t increases without bound? Explain.

77. Newton's Law of Cooling. This law states that the rate at which an object cools is proportional to the difference in temperature between the object and its surrounding medium. The temperature T of the object t hours later is given by

$$T = T_m + (T_0 - T_m)e^{-kt}$$

where T_m is the temperature of the surrounding medium and T_0 is the temperature of the object at $t = 0$. Suppose a bottle of wine at a room temperature of 72°F is placed in the refrigerator to cool before a dinner party. If the temperature in the refrigerator is kept at 40°F and $k = 0.4$, find the temperature of the wine, to the nearest degree, after 3 hours. (In Exercise 4-7 we will find out how to determine k.)

(A)