take note

Key Concepts Standard Form of a Polynomial Function

The **standard form of a polynomial function** arranges the terms by degree in descending numerical order.

A polynomial function $P(x)$ in standard form is

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

where n is a nonnegative integer and $a_n, \ldots, a_0$ are real numbers.

$$P(x) = 4x^3 + 3x^2 + 5x - 2$$

Cubic term | Quadratic term | Linear term | Constant term

You can classify a polynomial by its degree or by its number of terms. Polynomials of degrees zero through five have specific names, as shown in this table.

Degree	Name Using Degree	Polynomial Example	Number of Terms	Name Using Number of Terms
0	constant	5	1	monomial
1	linear	$x + 4$	2	binomial
2	quadratic	$4x^2$	1	monomial
3	cubic	$4x^3 - 2x^2 + x$	3	trinomial
4	quartic	$2x^4 + 5x^2$	2	binomial
5	quintic	$-x^5 + 4x^2 + 2x + 1$	4	polynomial of 4 terms

ONLINE PROBLEMS

Problem 1 Classifying Polynomials

Got It? Write each polynomial in standard form. What is the classification of each by degree? By number of terms?

a. $3x^3 - x + 5x^4$

b. $3 - 4x^5 + 2x^2 + 10$

Think

How do you write a polynomial in standard form?

A Practice Write each polynomial in standard form. Then classify it by degree and by number of terms.

1. $-4p + 3p + 2p^2$

2. $4x + 5x^2 + 8$

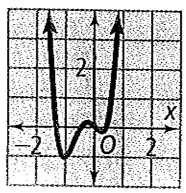
The degree of a polynomial function affects the shape of its graph and determines the maximum number of **turning points**, or places where the graph changes direction. It also affects the **end behavior**, or the directions of the graph to the far left and to the far right.

The table below shows you examples of polynomial functions and the four types of end behavior. The table also shows intervals where the functions are increasing and decreasing. A function is *increasing* when the *y*-values increase as *x*-values increase. A function is *decreasing* when the *y*-values decrease as *x*-values increase.

take note

Key Concepts Polynomial Functions

$y = 4x^4 + 6x^3 - x$

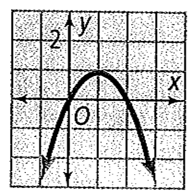


End behavior: up and up

Turning points: $(-1.07, -1.04)$, $(-0.27, 0.17)$, and $(0.22, -0.15)$

The function is decreasing when $x < -1.07$ and $-0.27 < x < 0.22$. The function is increasing when $-1.07 < x < -0.27$ and $x > 0.22$.

$y = -x^2 + 2x$

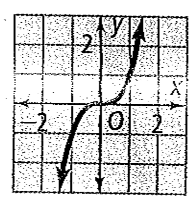


End behavior: down and down

Turning point: $(1, 1)$

The function is increasing when $x < 1$ and is decreasing when $x > 1$.

$y = x^3$

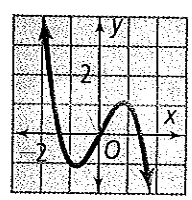


End behavior: down and up

No turning points

The function is increasing for all x.

$y = -x^3 + 2x$



End behavior: up and down

Turning Points: $(-0.82, -1.09)$ and $(0.82, 1.09)$

The function is decreasing when $x < -0.82$ and when $x > 0.82$. The function is increasing when $-0.82 < x < 0.82$.

You can determine the end behavior of a polynomial function of degree n from the leading term ax^n of the standard form.

End Behavior of a Polynomial Function With Leading Term ax^n

	n Even ($n \neq 0$)	n Odd
a Positive	Up and up	Down and up
a Negative	Down and down	Up and down

In general, the graph of a polynomial function of degree n ($n \geq 1$) has at most $n - 1$ turning points. The graph of a polynomial function of odd degree has an even number of turning points. The graph of a polynomial function of even degree has an odd number of turning points.

Problem 2 Describing End Behavior of Polynomial Functions

Got It? Consider the leading term of $y = -4x^3 + 2x^2 + 7$. What is the end behavior of the graph?

A Practice **Determine the end behavior of the graph of each polynomial function.**

3. $y = -14x^6 + 11x^5 - 11$

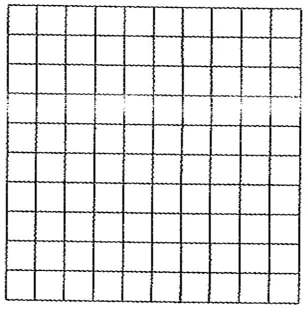
4. $y = x^3 - 14x - 4$

Problem 3 Graphing Cubic Functions

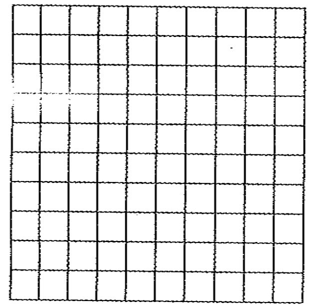
Plan
How can you graph a polynomial function?

Got It? What is the graph of each cubic function? Describe the graph.

a. $y = -x^3 + 2x^2 - x - 2$



b. $y = x^3 - 1$



Practice Describe the shape of the graph of each cubic function, including end behavior, turning points, and increasing/decreasing intervals.

5. $y = -9x^3 - 2x^2 + 5x + 3$

6. $y = 10x^3 + 9$

Recall that the FOIL method of multiplying binomials uses the Distributive Property to multiply each term of the first binomial by each term of the second binomial. You can extend this method to use the Distributive Property to multiply any two polynomials.

Problem 3 Multiplying Polynomials

Got It? What is each product?

a. $(a - 2b)(4a + 5b)$

b. $(5y^2 - 2y - 1)(y + 4)$

c. $(x - 1)(x + 1)(5x - 6)$

Practice Find each product.

5. $(x - 6)(x + 2)$

6. $(x + 4)(2x^2 - x + 8)$

Lesson Check

Do you know HOW?

Simplify each expression.

7. $(x^2 + 4x + 10) + (3x^2 - x - 12)$

8. $(3x^2 + 7x - 8) - (2x^2 + 2x - 6)$

9. $(x + 5)(x - 9)$

10. $(x - 2)(x^2 + 3x + 4)$

4-3 Polynomials, Linear Factors, and Zeros

F.IF.7.c Graph polynomial functions, identifying zeros . . . and showing end behavior.
Also A.SSE.1, A.APR.2, A.APR.3

Objectives To analyze the factored form of a polynomial
To write a polynomial function from its zeros

Solve It! Write your solution to the Solve It in the space below.

If $P(x)$ is a polynomial function, the solutions of the related polynomial equation $P(x) = 0$ are the zeros of the function.

Essential Understanding Finding the zeros of a polynomial function will help you factor the polynomial, graph the function, and solve the related polynomial equation.

In Chapter 3, you solved a quadratic equation of the form $x^2 + bx + c = 0$ by factoring. You wrote it using *linear factors* in the form $(x - r_1)(x - r_2) = 0$. Then you applied the Zero-Product Property to find the solutions $x = r_1$ and $x = r_2$. You can solve some polynomial equations $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ in much the same way.

Problem 1 Writing a Polynomial in Factored Form

Got It? What is the factored form of $x^3 - x^2 - 12x$?

Plan
How do you write the factored form of a polynomial?

Practice **Write each polynomial in factored form. Check by multiplication.**

1. $x^3 - 36x$

2. $9x^3 + 6x^2 - 3x$

take note

Key Concepts Roots, Zeros, and *x*-intercepts

The following are equivalent statements about a real number b and a polynomial

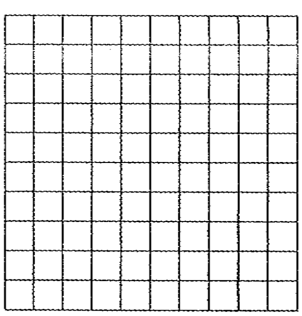
$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

- $x - b$ is a linear factor of the polynomial $P(x)$.
- b is a zero of the polynomial function $y = P(x)$.
- b is a root (or solution) of the polynomial equation $P(x) = 0$.
- b is an x-intercept of the graph of $y = P(x)$.

ONLINE PROBLEMS

Problem 2 Finding Zeros of a Polynomial Function

Got It? What are the zeros of $y = x(x - 3)(x + 5)$? Graph the function.



Practice Find the zeros of each function. Then graph the function.

3. $y = (x - 1)(x + 2)$

4. $y = x(x + 2)(x + 3)$

The Factor Theorem describes the relationship between the linear factors of a polynomial and the zeros of a polynomial.

take note

Theorem 102 Factor Theorem

The expression $x - a$ is a factor of a polynomial if and only if the value a is a zero of the related polynomial function.

Problem 3 Writing a Polynomial Function From Its Zeros

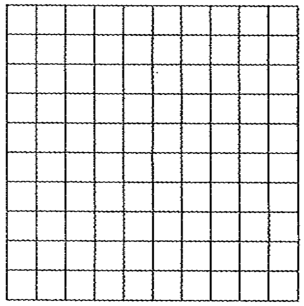
Plan

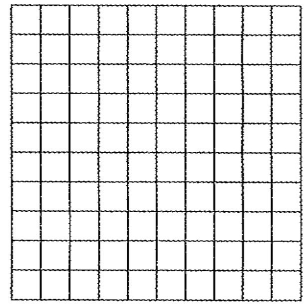
How can you use the zeroes to find the function?

Got It? **a.** What is a quadratic polynomial function with zeros 3 and -3?

b. What is a cubic polynomial function with zeros 3, 3, and -3?

c. Reasoning Graph both functions. How do the graphs differ? How are they similar?





Practice **Write a polynomial function in standard form with the given zeros.**

5. $x = 1, -1, -2$

6. $x = -1, -2, -3, -4$

You can write the polynomial functions in Problem 3 in factored form as $f(x) = (x + 2)(x - 2)(x - 3)$ and $g(x) = (x + 2)^2(x - 2)(x - 3)$. In $g(x)$ the repeated linear factor $x + 2$ makes -2 a **multiple zero**.

In particular, since the linear factor $x + 2$ appears twice, you can say that -2 is a zero of **multiplicity** 2. In general, *a is a zero of multiplicity n* means that $x - a$ appears n times as a factor.

take note

Key Concepts How Multiple Zeros Affect a Graph

If a is a zero of multiplicity n in the polynomial function $y = P(x)$, then the behavior of the graph at the x-intercept a will be close to linear if $n = 1$, close to quadratic if $n = 2$, close to cubic if $n = 3$, and so on.

ONLINE PROBLEMS

Problem 4 Finding the Multiplicity of a Zero

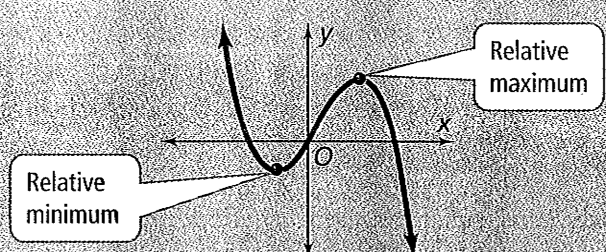
Got It? What are the zeros of $f(x) = x^3 - 4x^2 + 4x$? What are their multiplicities? How does the graph behave at these zeros?

A Practice **Find the zeros of each function. State the multiplicity of multiple zeros.**

7. $y = 2x^3 + x^2 - x$

8. $y = (x + 1)^2(x - 1)(x - 2)$

If the graph of a polynomial function has several turning points, the function can have a **relative maximum** and a **relative minimum**. A relative maximum is the value of the function at an up-to-down turning point. A relative minimum is the value of the function at a down-to-up turning point.



Problem 5 Identifying a Relative Maximum and Minimum

Got It? What are the relative maximum and minimum of $f(x) = 3x^3 + x^2 - 5x$?

Practice **Find the relative maximum and relative minimum of the graph of each function.**

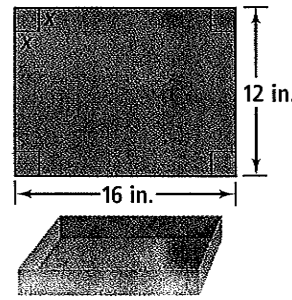
9. $f(x) = -x^3 + 16x^2 - 76x + 96$

10. $f(x) = x^3 - 7x^2 + 7x + 15$

Problem 6 Using a Polynomial Function to Maximize Volume

Got It? What is the maximum volume of the camera in Problem 6, if the sum of the dimensions is at most 4 inches?

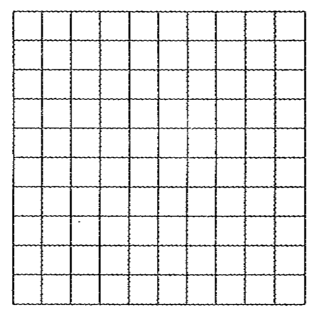
A Practice 11. **Metalwork** STEM A metalworker wants to make an open box from a sheet of metal, by cutting equal squares from each corner as shown.



a. Write expressions for the length, width, and height of the open box.

b. Use your expressions from part (a) to write a function for the volume of the box. (*Hint:* Write the function in factored form.)

c. Graph the function. Then find the maximum volume of the box and the side length of the cut-out squares that generates this volume.



Lesson Check

Do you know HOW?

Find the zeros of each function.

12. $y = x(x - 6)$

13. $y = (x + 4)(x - 5)$

14. $y = (x + 12)(x - 9)(x - 7)$

15. Write a polynomial function in standard form with zeros -1, 1, and 0.

Do you UNDERSTAND?

MATHEMATICAL PRACTICES

16. Vocabulary Write a polynomial function h in standard form that has 3 and -5 as zeros of multiplicity 2.

17. Error Analysis Your friend says that to write a function that has zeros 3 and -1, you should multiply the two factors $(x + 3)$ and $(x - 1)$ to get $f(x) = x^2 + 2x - 3$. Describe and correct your friend's error.

More Practice and Problem-Solving Exercises

MATHEMATICAL PRACTICES

B Apply

Write each function in factored form. Check by multiplication.

18. $y = 3x^3 - 27x^2 + 24x$ **19.** $y = -2x^3 - 2x^2 + 40x$ **20.** $y = x^4 + 3x^3 - 4x^2$

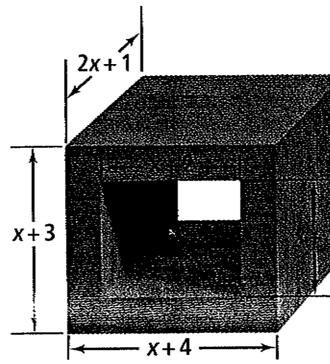
21. Think About a Plan A storage company needs to design a new storage box that has twice the volume of its largest box. Its largest box is 5 ft long, 4 ft wide, and 3 ft high. The new box must be formed by increasing each dimension by the same amount. Find the increase in each dimension.

- How can you write the dimensions of the new storage box as polynomial expressions?
- How can you use the volume of the current largest box to find the volume of the new box?

22. **Carpentry** A carpenter hollowed out the interior of a block of wood as shown at the right.

   a. Express the volume of the original block and the volume of the wood removed as polynomials in factored form.

   b. What polynomial represents the volume of the wood remaining?



23. **Geometry** A rectangular box is $2x + 3$ units long, $2x - 3$ units wide, and $3x$ units high. What is its volume, expressed as a polynomial?

24. **Measurement** The volume in cubic feet of a CD holder can be expressed as $V(x) = -x^3 - x^2 + 6x$ or, when factored, as the product of its three dimensions. The depth is expressed as $2 - x$. Assume that the height is greater than the width.

   a. Factor the polynomial to find linear expressions for the height and the width.

   b. Graph the function. Find the x-intercepts. What do they represent?

   c. What is a realistic domain for the function?

   d. What is the maximum volume of the CD holder?

Find the relative maximum, relative minimum, and zeros of each function.

25. $y = 2x^3 - 23x^2 + 78x - 72$ 26. $y = 8x^3 - 10x^2 - x - 3$ 27. $y = (x + 1)^4 - 1$

28. **Open-Ended** Write a polynomial function with the following features: it has three distinct zeros; one of the zeros is 1; another zero has a multiplicity of 2.

29. **Writing** Explain how the graph of a polynomial function can help you factor the polynomial.

For each function, determine the zeros. State the multiplicity of any multiple zeros.

30. $f(x) = x^3 - 36x$ 31. $y = (x + 1)(x - 4)(3 - 2x)$ 32. $y = (x + 7)(5x + 2)(x - 6)^2$

Challenge

33. Find a fourth-degree polynomial function with zeros 1, -1, i, and $-i$. Write the function in factored form.

34. a. Compare the graphs of $y = (x + 1)(x + 2)(x + 3)$ and $y = (x - 1)(x - 2)(x - 3)$. What transformation could you use to describe the change from one graph to the other?

   b. Compare the graphs of $y = (x + 1)(x + 3)(x + 7)$ and $y = (x - 1)(x - 3)(x - 7)$. Does the transformation that you chose in part (a) still hold true? Explain.

   c. **Make a Conjecture** What transformation could you use to describe the effect of changing the signs of the zeros of a polynomial function?

Practice Determine whether each binomial is a factor of $x^3 + 4x^2 + x - 6$.

3. $x + 2$

4. $x - 3$

Synthetic division simplifies the long-division process for dividing by a linear expression $x - a$. To use synthetic division, write the coefficients (including zeros) of the polynomial in standard form. Omit all variables and exponents. For the divisor, reverse the sign (use a). This allows you to add instead of subtract throughout the process.

Problem 3 Using Synthetic Division

Got It? Use synthetic division to divide $x^3 - 57x + 56$ by $x - 7$. What are the quotient and remainder?

Think

To divide by $x - 7$ what number do you use for the synthetic divisor?

Practice Divide using synthetic division.

5. $(x^3 - 3x^2 - 5x - 25) \div (x - 5)$

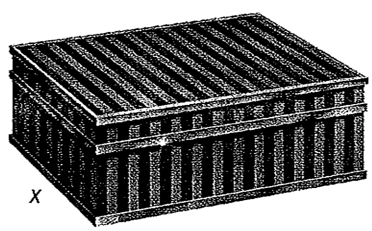
6. $(6x^2 - 8x - 2) \div (x - 1)$

Problem 4 Using Synthetic Division to Solve a Problem

Got It? If the polynomial $x^3 + 6x^2 + 11x + 6$ expresses the volume, in cubic inches, of the box, and the width is $(x + 1)$ in., what are the dimensions of the box?

Practice 7. Use the factor $x + 3$ and synthetic division to completely factor the function $y = x^3 - 4x^2 - 9x + 36$.

8. **Geometry** The volume, in cubic inches, of the decorative box shown can be expressed as the product of the lengths of its sides as $V(x) = x^3 + x^2 - 6x$. What linear expressions with integer coefficients represent the length and height of the box?



The Remainder Theorem provides a quick way to find the remainder of a polynomial long-division problem.

take note

Theorem 103 The Remainder Theorem

If you divide a polynomial $P(x)$ of degree $n \geq 1$ by $x - a$, then the remainder is $P(a)$.

Here's Why It Works When you divide polynomial $P(x)$ by $D(x)$, you find $P(x) = D(x)Q(x) + R(x)$.

$P(x) = (x - a)Q(x) + R(x)$	Substitute $(x - a)$ for $D(x)$.
$P(a) = (a - a)Q(a) + R(a)$	Evaluate $P(a)$. Substitute a for x.
$= R(a)$	Simplify.

Problem 5 Evaluating a Polynomial

Got It? Given that $P(x) = x^5 - 3x^4 - 28x^3 + 5x + 20$, what is $P(-4)$?

A Practice Use synthetic division and the Remainder Theorem to find $P(a)$.

9. $P(x) = 6x^3 - x^2 + 4x + 3$; $a = 3$

10. $P(x) = 2x^3 - x^2 + 10x + 5$; $a = \frac{1}{2}$

Lesson Check

Do you know HOW?

Divide using any method.

11. $(2x^2 + 7x + 11) \div (x + 2)$

12. $(x^3 + 5x^2 + 11x + 15) \div (x + 3)$

13. $(x^3 - x^2 - 4x + 4) \div (x - 2)$

14. $(4x^3 + 21x^2 - x - 24) \div (x + 5)$

15. $(9x^3 - 15x^2 + 4x) \div (x - 3)$

17. Reasoning In the statements below, r and s represent integers. Is each statement *always, sometimes,* or *never* true? Explain.

a. A root of the equation $3x^3 + rx^2 + sx + 8 = 0$ could be 5.

b. A root of the equation $3x^3 + rx^2 + sx + 8 = 0$ could be -2.

18. Error Analysis A student claims that $-4i$ is the only imaginary root of a polynomial equation that has real coefficients. What is the student's mistake?

More Practice and Problem-Solving Exercises

MATHEMATICAL PRACTICES

B Apply

Find all rational roots for $P(x) = 0$.

19. $P(x) = 2x^3 - 5x^2 + x - 1$

20. $P(x) = 6x^4 - 13x^3 + 13x^2 - 39x - 15$

21. $P(x) = 7x^3 - x^2 - 5x + 14$

22. $P(x) = 3x^4 - 7x^3 + 10x^2 - x + 12$

23. $P(x) = 6x^4 - 7x^2 - 3$

24. $P(x) = 2x^3 - 3x^2 - 8x + 12$

ACTIVITY LAB

Use With Lesson 4-7

Graphing Polynomials Using Zeros

A.APR.3 Identify zeros of polynomials when suitable factorizations are available and use the zeros to construct a rough graph. Also F.IF.7.c

In this activity you will learn how to sketch the graph of a polynomial function by using the zeros, turning points, and end behavior.

Example

Sketch the graph of the function $f(x) = (x - 3)(x + 1)(x - 2)$.

Step 1 Identify the zeros and plot them on a coordinate grid.

$$f(x) = (x - 3)(x + 1)(x - 2)$$

$$0 = (x - 3)(x + 1)(x - 2)$$

$$0 = x - 3 \quad \text{or} \quad 0 = x + 1 \quad \text{or} \quad 0 = x - 2$$

$$3 = x \qquad\qquad -1 = x \qquad\qquad 2 = x$$

The function has zeros at (3, 0), (−1, 0), and (2, 0).

Step 2 Determine whether the polynomial is positive or negative over each interval.

To determine whether $f(x)$ is positive or negative over the interval $x < -1$, choose an x-value within the interval. Let $x = -2$. Then evaluate $f(-2)$.

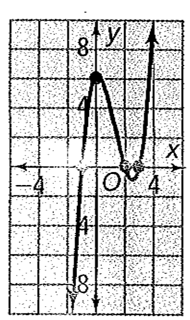
$$f(-2) = (-2 - 3)(-2 + 1)(-2 - 2) = (-5)(-1)(-4) = -20$$

$f(x)$ is negative over the interval $x < -1$.

Repeat the process for the intervals $-1 < x < 2$, $2 < x < 3$, and $x > 3$.

Interval	x	$f(x)$
$x < -1$	−2	−20
$-1 < x < 2$	0	6
$2 < x < 3$	2.5	−0.875
$x > 3$	4	10

Step 3 Sketch the graph.



Exercises

Sketch a graph of each function. Check your answer using a graphing calculator.

1. $h(x) = (x + 6)(x - 7)$

2. $g(x) = (x + 1)(x - 3)(x - 5)$

3. $p(x) = x(x + 4)(x - 4)$

4. $h(x) = (x + 2)(x - 3)(x + 1)(x - 1)$

5. $f(x) = x^4 - 8x^2 + 16$

6. $k(x) = x^4 - 10x^2 + 9$

take note

Key Concept The $n + 1$ Point Principle

For any set of $n + 1$ points in the coordinate plane that pass the vertical line test, there is a unique polynomial of degree at most n that fits the points perfectly.

This principle confirms that any two points determine a unique line. Three points that are not on a line determine a unique parabola. Four points that are not on a line or a parabola determine a unique cubic, and so forth.

ONLINE PROBLEMS

Problem 1 Using A Polynomial Function to Model Data

Plan

How would the $n + 1$ Point Principle be helpful in solving this?

Got It? What polynomial function has a graph that passes through the four points $(-2, 1)$, $(0, 5)$, $(2, 9)$, and $(3, 36)$?

A Practice Find a polynomial function whose graph passes through each set of points.

1. $(7, 13)$, $(10, -11)$, and $(0, 4)$

2. $(-1, 9)$, $(0, 6)$, $(1, 5)$, and $(2, 18)$

Problem 2 Modeling Data

Got It? STEM Use the linear model $f(x) = 0.128x + 10.36$ from Problem 2. Estimate Wisconsin milk production in 1995.

Think What value of x should you use to predict the milk production?

Practice **For each set of data, what cubic model best fits the data? Use the model to estimate a value for the year 2012.**

3. **U.S. Federal Spending**

Year	Total (billions $)
1965	630
1980	1,300
1995	1,950
2005	2,650

4. **World Population**

Year	Average Growth Rate (%)
1972	1.96
1982	1.73
1992	1.5
2002	1.22

14. **Reasoning** Is it possible to create a cubic function that passes through (0, 0), (−1, 1), (−2, 2), and (−3, 9)? Explain.

15. **Writing** The R^2 value for a quartic model is 0.94561. The R^2 value for a cubic model of the same data is 0.99817. Which model seems to show a better fit? Explain.

More Practice and Problem-Solving Exercises

MATHEMATICAL PRACTICES

B Apply

Find a cubic and a quartic model for each set of values. Explain why one models the data better.

16.

x	−2	−1	0	2	3
y	−25	−4	3	23	40

17.

x	−2	−1	0	1	2
y	−65	−14	−4	2	90

Find a polynomial function whose graph passes through the points.

18. (−14, 14), (−10, 0), (0, −1), (8, 0), and (12, 4)

19. (−3, −50), (−2, −4), (−1, 10), (0, 7), and (2, −23)

20. **Think About a Plan** The table at the right shows the amount of carbon dioxide in the Earth's atmosphere for selected years. Predict the amount of carbon dioxide in the Earth's atmosphere in 2022. How confident are you in your prediction?
- How can you plot the data? (*Hint:* Let *x* equal the years after 1900.)
- What polynomial model should you use?

Year	CO^2 in atmosphere (ppm)
1968	324.14
1983	343.91
1998	367.68
2003	376.68
2008	385.60

SOURCE: The Weather Channel

Find a cubic model for each set of values. Then use the regression coefficient of each model to determine whether the model is a good fit.

21. (−5, −60), (−1, −5), (0, 0.5), (1, 8), (5, 17), (10, 32)

22. (8, −101), (−1, 10), (−8, 47), (−10, 59)

23. Air Travel The table shows the percent of on-time flights for selected years. Find a polynomial function to model the data. Use 1998 as Year 0.

Year	1998	2000	2002	2004	2006
On-Time Flights (%)	77.20	72.59	82.14	78.08	75.45

SOURCE: U.S. Bureau of Transportation Statistics

24. Writing Explain two ways to find a polynomial function to model a given set of data.

25. Error Analysis The table at the right shows the number of students enrolled in a high school personal finance course. A student says that a cubic model would best fit the data based on the $n + 1$ Point Principle. Explain why a quadratic model might be more appropriate.

Year	Number of Students Enrolled
2000	50
2004	65
2008	94
2010	110

26. Compare and Contrast The table shows the United States gross domestic product for selected years. Construct curves using cubic regression and quartic regression to model the data. Which curve seems most likely to model gross domestic product over the years?

Year	1960	1970	1980	1990	2000
GDP (billions $)	526.4	1038.5	2789.5	5803.1	9817.0

27. The table below shows the percentage of the U.S. labor force in unions for selected years between 1955 and 2005.

Year	1955	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
%	33.2	31.4	28.4	27.3	25.5	21.9	18.0	16.1	14.9	13.5	12.5

a. What is the average rate of change between 1955 and 1965? Between 1975 and 1985?

b. Make a scatter plot of the data. Which kind of polynomial model seems to be most appropriate?

c. Use a graphing calculator to find the type of model from part (b).

d. Use the model you found in part (c) to predict the percent of the labor force in unions in the year 2020.

e. Reasoning Do you have much confidence in this prediction? Explain.