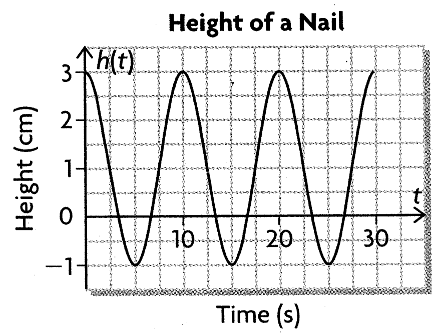
# 6.6 Investigating Models of Sinusoidal Functions

**GOAL**

Determine the equation of a sinusoidal function from a graph or a table of values.

## *LEARN ABOUT the Math*

A nail located on the circumference of a water wheel is moving as the current pushes on the wheel. The height of the nail in terms of time can be modelled by the graph shown.




**?** How can you determine the equation of a sinusoidal function from its graph?

### EXAMPLE 1 Representing a sinusoidal graph using the equation of a function

Determine an equation of the given graph.

### Sasha's Solution

Horizontal compression factor: $k$

$$\text{period} = \frac{360}{|k|}$$

The period is 10 s.

$k > 0$, so $|k| = k$

I calculated the period, equation of the axis, and amplitude. Then I figured out how they are related to different transformations.

The period is 10 s since the peaks are 10 units apart. The horizontal stretch or compression factor $k$ had to be positive because the graph was not reflected in the $y$-axis. I used the formula relating $k$ to the period.

$$10 = \frac{360}{k}$$

$$k = \frac{360}{10}$$

$$k = 36$$

The graph was compressed by a factor of $\frac{1}{36}$.

Vertical translation: $c$

The axis is halfway between the maximum, 3, and the minimum, −1. This gave me the vertical translation and the value of $c$.

$$\text{equation of the axis} = \frac{\max + \min}{2}$$

$$= \frac{3 + (-1)}{2}$$

$$= 1 \text{ (vertical translation)}$$

$$c = 1$$

Vertical stretch: $a$

I calculated the amplitude by taking the maximum value, 3, and subtracting the axis, 1. Since the amplitude of $y = \cos x$ is 1, and the amplitude of this graph is 2, the vertical stretch is 2.

$a = 2$

Base graph: $y = \cos x$

The cosine curve is easier to use for my equation since the graph has its maximum on the $y$-axis, just as this graph does. This means that for a cosine curve, there isn't any horizontal translation, so $d = 0$.
I found the equation of the function by substituting the values I calculated into $f(x) = a\cos(k(x - d)) + c$.

As a cosine curve:

$y = 2\cos(36x)^\circ + 1$

I could have used the sine function instead.

As a sine curve:

$y = 2\sin(36(x - 7.5))^\circ + 1$

A sine curve increases from a $y$-value of 0 at $x = 0$.

On this graph, that happens at 7.5. This means that, for a sine curve, there is a horizontal translation of 7.5, so $c = 7.5$.

For both functions, the domain is restricted to $x \geq 0$ because it represents the time elapsed.

## Reflecting

**A.** Tanya says that another possible equation of the sinusoidal function created by Sasha is $y = 2\cos(36(x - 10))^\circ + 1$. Is she correct? Why or why not?

**B.** If the period on the original water wheel graph is changed from 10 to 20, what would be the new equation of the sinusoidal function?

**C.** If the maximum value on the original water wheel graph is changed from 3 to 5, what would be the new equation of the sinusoidal function?

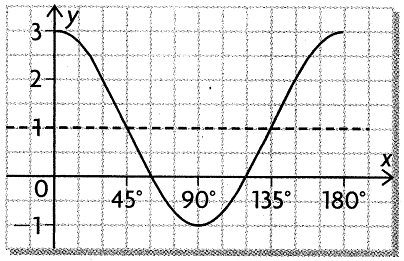
**D.** If the speed of the current increases so that the water wheel spins twice as fast, what would be the equation of the resulting function?

## *APPLY* the Math

### EXAMPLE 2 Connecting the equation of a sinusoidal function to its features

A sinusoidal function has an amplitude of 2 units, a period of 180°, and a maximum at (0, 3). Represent the function with an equation in two different ways.

### Rajiv's Solution




The graph has a maximum at (0, 3) and a period of 180°, so the next maximum would be at (180, 3).

A minimum would be halfway between the two maximums.

Since the amplitude is 2, and $2 - 3 = -1$, the minimum would have to be at (90°, −1).

Vertical translation: $c = 1$

Vertical stretch: $a$

The equation of the axis gave me the vertical translation. Since the equation is $y = 1$ instead of $y = 0$, there was a vertical translation of 1.

amplitude $= 3 - 1 = 2$

$a = 2$

The amplitude gives me the vertical stretch.

Horizontal compression: $k$

$$\text{period} = \frac{360°}{|k|}$$

$$180° = \frac{360°}{k}$$

$$k = \frac{360°}{180°}$$

$$k = 2$$

The period is 180°, so there has been a horizontal compression. Since there was no horizontal reflection, $k > 0$. To find $k$, I took 360° and divided it by the period.

Compression factor is $\frac{1}{2}$.

For a cosine curve:

No horizontal translation so

$d = 0$

Equation:

$y = 2\cos(2x) + 1$

Cosine curves have a maximum at $x = 0$, unless they've been translated horizontally. This curve starts at its maximum, so there would be no horizontal translation with a cosine function as a model.

I found the equation of the function by substituting the values into $f(x) = a\cos(k(x - d)) + c$.

For a sine curve:

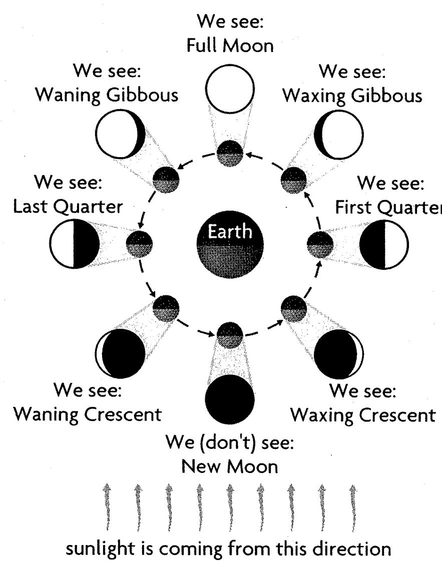
horizontal translation = 135°

$y = 2\sin(2(x - 135°)) + 1$

The equation of the axis of this cosine curve is $y = 1$. On this cosine curve, the point (135°, 1) corresponds to the start of the cycle of the sine function. The sine curve with the same period and axis as this cosine curve has the equation $y = 2\sin(2x) + 1$, but its starting point is (0°, 1). This means the function $y = 2\sin(2x) + 1$ must be translated horizontally to the right by 135°, so $d = 135°$.

## EXAMPLE 3 Connecting data to the algebraic model of a sinusoidal function

The Moon is always half illuminated by the Sun. How much of the Moon we see depends on where it is in its orbit around Earth. The table shows the proportion of the Moon that was visible from Southern Ontario on days 1 to 74 in the year 2006.



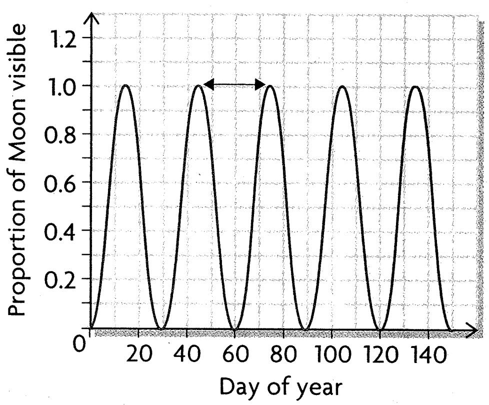

| Day of Year | 1 | 4 | 7 | 10 | 14 | 20 | 24 | 29 | 34 |
|---|---|---|---|---|---|---|---|---|---|
| Proportion of Moon Visible | 0.02 | 0.22 | 0.55 | 0.83 | 1.00 | 0.73 | 0.34 | 0.00 | 0.28 |

| Day of Year | 41 | 44 | 48 | 53 | 56 | 59 | 63 | 70 | 74 |
|---|---|---|---|---|---|---|---|---|---|
| Proportion of Moon Visible | 0.92 | 1.00 | 0.86 | 0.41 | 0.12 | 0.00 | 0.23 | 0.88 | 1.00 |

a) Determine the equation of the sinusoidal function that models the proportion of visible Moon in terms of time.
b) Determine the domain and range of the function.
c) Use the equation to determine the proportion of the Moon that is visible on day 110.

### Rosalie's Solution

a) **Cycle of the Proportion of the Moon Visible**




I plotted the data. When I drew the curve, the graph looked like a sinusoidal function.

The maximum value was 1, and the minimum value 0.

The graph repeats every 30 days, so the period must be 30 days.

I figured out some of the important features of the sinusoidal function.

Vertical translation: $c$

Equation of the axis is $y = 0.5$.

$c = 0.5$

The axis is halfway between the maximum of 1 and the minimum of 0.

Vertical stretch: $a$

$$\text{amplitude} = \frac{(1-0)}{2}$$

The amplitude is the vertical distance between the maximum and the axis. In this case, it is 0.5, or $\frac{1}{2}$.

$$= 0.5 \quad \text{or} \quad \frac{1}{2}$$

$$a = 0.5$$

Horizontal compression: $k$

$$\text{period} = \frac{360}{|k|}$$

I used the period to get the compression.

$k > 0$, so $|k| = k$

$$30 = \frac{360°}{k}$$

$$k = \frac{360°}{30}$$

$$k = 12$$

Horizontal translation: $d$

A sine curve or a cosine curve will work. I used the cosine curve. The horizontal translation is equal to the $x$-coordinate of a maximum, since $y = \cos x$ has a maximum at $x = 0$. I chose the $x$-coordinate of the maximum closest to the origin, $x = 14$.

Using a cosine curve:

$d = 14$

$$y = \frac{1}{2}\cos(12(x - 14)°) + 0.5$$

I put the information together to get the equation.

**b)** domain: $\{x \in \mathbf{R} \mid 0 \leq x \leq 365\}$

range: $\{y \in \mathbf{R} \mid 0 \leq y \leq 1\}$

The domain is only the non-negative values of $x$ up to 365, since they are days of the year. The range is 0 to 1.

**c)** $y = \frac{1}{2}\cos(12(x - 14)°) + 0.5$

Since $x$ represents the time in days, I substituted 110 for $x$ in the equation to calculate the amount of the Moon visible at that time. Then I solved for $y$.

$$= \frac{1}{2}\cos(12(110 - 14)°) + 0.5$$

$$= \frac{1}{2}\cos(1152)° + 0.5$$

$$\doteq \frac{1}{2}(0.3090) + 0.5$$

$$= 0.65$$

On day 110, 65% of the Moon is exposed.

## In Summary

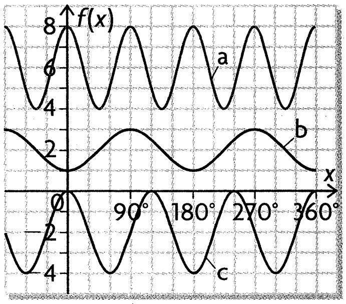
### Key Idea

- If you are given a set of data and the corresponding graph is a sinusoidal function, then you can determine the equation by calculating the graph's period, amplitude, and equation of the axis. This information will help you determine the values of $k$, $a$, and $c$, respectively, in the equations $g(x) = a\sin(k(x - d)) + c$ and $h(x) = a\cos(k(x - d)) + c$. The value of $d$ is determined by estimating the required horizontal shift (left or right) compared with the graph of the sine or cosine curve.

### Need to Know

- If the graph begins at a maximum value, it may be easier to use the cosine function as your model.
- The domain and range of a sinusoidal model may need to be restricted for the situation you are dealing with.

## *CHECK Your Understanding*




1. Determine an equation for each sinusoidal function at the right.

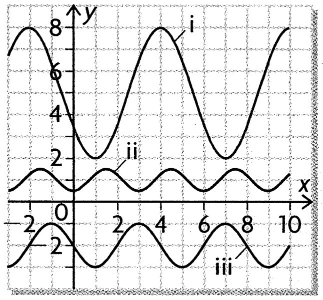
2. Determine the function that models the data in the table and does not involve a horizontal translation.

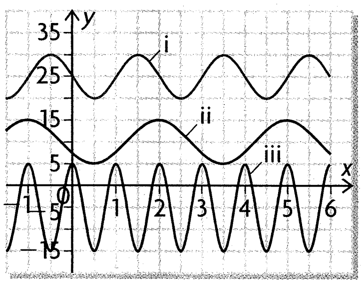
| $x$ | 0° | 45° | 90° | 135° | 180° | 225° | 270° |
|---|---|---|---|---|---|---|---|
| $y$ | 9 | 7 | 5 | 7 | 9 | 7 | 5 |

3. A sinusoidal function has an amplitude of 4 units, a period of 120°, and a maximum at (0, 9). Determine the equation of the function.

## *PRACTISING*

4. Determine the equation for each sinusoidal function.

a)



b)


5. For each table of data, determine the equation of the function that is the simplest model.

a)

| $x$ | 0° | 30° | 60° | 90° | 120° | 150° | 180° |
|---|---|---|---|---|---|---|---|
| $y$ | 3 | 2 | 1 | 2 | 3 | 2 | 1 |

b)

| $x$ | −180° | 0° | 180° | 360° | 540° | 720° | 900° |
|---|---|---|---|---|---|---|---|
| $y$ | 17 | 13 | 17 | 21 | 17 | 13 | 17 |

c)

| $x$ | −120° | −60° | 0° | 60° | 120° | 180° | 240° |
|---|---|---|---|---|---|---|---|
| $y$ | −4 | −7 | −4 | −1 | −4 | −7 | −4 |

d)

| $x$ | −20° | 10° | 40° | 70° | 100° | 130° | 160° |
|---|---|---|---|---|---|---|---|
| $y$ | 2 | 5 | 2 | −1 | 2 | 5 | 2 |

6. **K** Determine the equation of the cosine function whose graph has each of the following features.

| | Amplitude | Period | Equation of the Axis | Horizontal Translation |
|---|---|---|---|---|
| a) | 3 | 360° | $y = 11$ | 0° |
| b) | 4 | 180° | $y = 15$ | 30° |
| c) | 2 | 40° | $y = 0$ | 7° |
| d) | 0.5 | 720° | $y = -3$ | −56° |

7. A sinusoidal function has an amplitude of 6 units, a period of 45°, and a minimum at (0, 1). Determine an equation of the function.

8. **A** The table shows the average monthly high temperature for one year in Kapuskasing, Ontario.

| Time (months) | J | F | M | A | M | J | J | A | S | O | N | D |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Temperature (°C) | −18.6 | −16.3 | −9.1 | 0.4 | 8.5 | 13.8 | 17.0 | 15.4 | 10.3 | 4.4 | −4.3 | −14.8 |

a) Draw a scatter plot of the data and the curve of best fit. Let January be month 0.

b) What type of model describes the graph? Why did you select that model?

c) Write an equation for your model. Describe the constants and the variables in the context of this problem.

d) What is the average monthly temperature for month 20?

9. The table shows the velocity of air of Nicole's breathing while she is at rest.

| Time (s) | 0 | 0.25 | 0.5 | 0.75 | 1.0 | 1.25 | 1.5 | 1.75 | 2 | 2.25 | 2.5 | 2.75 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Velocity (L/s) | 0 | 0.22 | 0.45 | 0.61 | 0.75 | 0.82 | 0.85 | 0.83 | 0.74 | 0.61 | 0.43 | 0.23 | 0 |

a) Explain why breathing is an example of a periodic function.
b) Graph the data, and determine an equation that models the situation.
c) Using a graphing calculator, graph the data as a scatter plot. Enter your equation and graph. Comment on the closeness of fit between the scatter plot and the graph.
d) What is the velocity of Nicole's breathing at 6 s? Justify.
e) How many seconds have passed when the velocity is 0.5 L/s?

**Tech | *Support***

For help creating a scatter plot using a graphic calculator, see Technical Appendix, B-11.

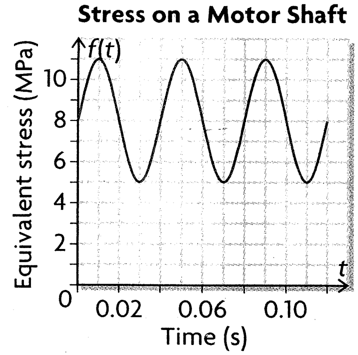
10. The table shows the average high monthly temperature for three cities: Athens, Lisbon, and Moscow.

| Time (month) | J | F | M | A | M | J | J | A | S | O | N | D |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Athens (°C) | 12 | 13 | 15 | 19 | 24 | 30 | 33 | 32 | 28 | 23 | 18 | 14 |
| Lisbon (°C) | 13 | 14 | 16 | 18 | 21 | 24 | 26 | 27 | 24 | 21 | 17 | 14 |
| Moscow (°C) | −9 | −6 | 0 | 10 | 19 | 21 | 23 | 22 | 16 | 9 | 1 | −4 |

a) Graph the data to show that temperature is a function of time for each city.
b) Write the equations that model each function.
c) Explain the differences in the amplitude and the vertical translation for each city.
d) What does this tell you about the cities?

11. **C** The relationship between the stress on the shaft of an electric motor and time can be modelled with a sinusoidal function. (The units of stress are megapascals (MPa).)
a) Determine an equation of the function that describes the equivalent stress in terms of time.
b) What do the peaks of the function represent in this situation?
c) How much stress was the motor undergoing at 0.143 s?

**Stress on a Motor Shaft**



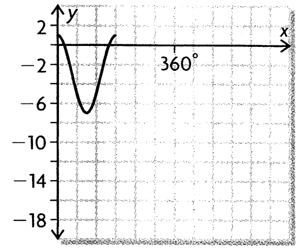

12. **T** Describe a procedure for writing the equation of a sinusoidal function based on a given graph.

## Extending

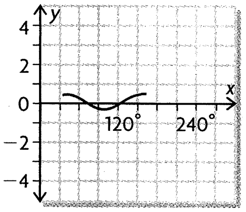
13. The diameter of a car's tire is 60 cm. While the car is being driven, the tire picks up a nail. How high above the ground is the nail after the car has travelled 1 km?

14. Matthew is riding a Ferris wheel at a constant speed of 10 km/h. The boarding height for the wheel is 1 m, and the wheel has a radius of 7 m. What is the equation of the function that describes Matthew's height in terms of time, assuming Matthew starts at the highest point on the wheel?

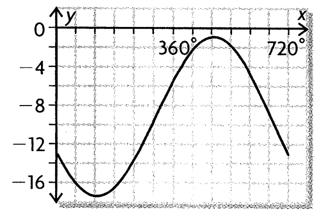
d) vertical stretch by a factor of 4, horizontal compression by a factor of $\frac{1}{2}$, and vertical translation 3 units down




e) vertical compression by a factor of $\frac{1}{2}$, horizontal compression by a factor of $\frac{1}{3}$, and horizontal translation 40° to the right




f) vertical stretch by a factor of 8, reflection in the $x$-axis, horizontal stretch by a factor of 2, horizontal translation 50° to the left, and vertical translation 9 units down




**8.**

| | X min | X max | Y min | Y max |
|---|---|---|---|---|
| a) | 0° | 180° | 5 | 7 |
| b) | 0° | 720° | 15 | 25 |
| c) | 0° | 4° | 75 | 89 |
| d) | 0° | 1° | −27.5 | −26.5 |

**9.** a) period: 1.2 s; one heart beat
b) $\{P \in \mathbf{R} \mid 80 \leq P \leq 120\}$, maximum blood pressure of 120, minimum blood pressure of 80

**10.** Answers may vary. For example,
a) $y = 4 \sin\left(\frac{1}{2}x\right) + 3$
b) $y = 4 \cos\left(\frac{1}{2}x - 90\right) + 3$

**11.** Reflection in x-axis, vertical compression of $\frac{1}{2}$, vertical translation 30 up, horizontal compression of $\frac{1}{120}$.

**12.** horizontal translation: $-45°$

**13.** a) The number of hours of daylight increases to a maximum and decreases to a minimum in a regular cycle as Earth revolves around the Sun.
b) Mar. 21: 12 h; Sept. 21: 12 h, spring and fall equinoxes
c) June 21: 16 h; Dec. 21: 8 h, longest and shortest days of year; summer and winter solstices
d) 12 is the axis of the curve representing half the distance between the maximum and minimum hours of daylight.

## Lesson 6.6, pp. 391–393

**1.** Answers may vary. For example,
a) $y = 2\cos(4x) + 6$
b) $y = \cos(2(x - 90°)) + 2$
c) $y = 2\cos(3x) - 2$

**2.** $y = 2\cos(2x) + 7$

**3.** Answers may vary. For example, $y = 4\cos(3x) + 5$

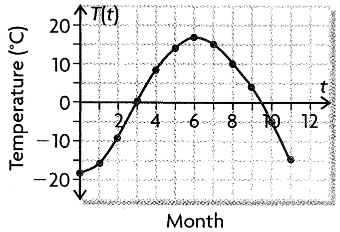
**4.** Answers may vary. For example,
a) i) $y = 3\cos(60(x - 4°)) + 5$;
ii) $y = -0.5\cos(120x) + 1$;
iii) $y = \cos(90(x - 3°)) - 2$
b) i) $y = 5\cos(180(x - 1.5°)) + 25$;
ii) $y = 5\cos(120(x - 2°)) + 10$;
iii) $y = 10\cos(360x) - 5$

**5.** a) $y = \cos(3x) + 17$
b) $y = -4\cos(0.5x) + 2$
c) $y = 3\sin(1.5x) - 4$
d) $y = 3\cos(3(x - 10°)) + 2$

**6.** a) $y = 3\cos x + 11$
b) $y = 4\cos(2(x - 30°)) + 15$
c) $y = 2\cos(9(x - 7°))$
d) $y = 0.5\cos\left(\left(\frac{1}{2}\right)(x + 56°)\right) - 3$

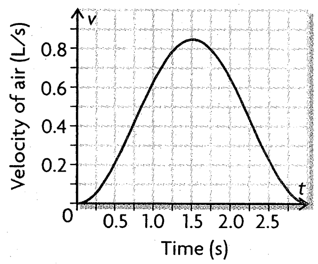
**7.** Answers may vary. For example, $y = -6\cos(8x) + 7$

**8.** a)



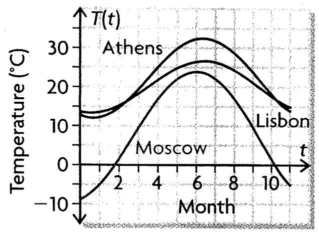

b) sinusoidal model because it changes with a cyclical pattern over time; the data is wave-shaped.
c) $T(t) = -17.8 \cos 30t - 0.8$
d) Answers may vary. For example, 8.1°C or 10.3°C using the chart.

**9.** a) The respiratory cycle is an example of a periodic function because we inhale, rest, exhale, rest, inhale, and so on in a cyclical pattern.
b) $v = -0.425 \cos(120t)° + 0.425$




c) The fit is somewhat close.
d) 0 L/s; period is 3; troughs occur at 0, 3, and 6 s
e) Answers may vary. For example, 0.8 s and 2.2 s from model or 0.6 s and 2.4 s from chart.

**10.** a)




b) Athens: $T(t) = -10.5 \cos 30t + 22.5$;
Lisbon: $T(t) = -7 \cos 30\text{t} + 20$;
Moscow: $T(t) = -16 \cos 30t + 7$

c) Latitude affects average temperature as well as maximum and minimum temperatures.

d) Athens and Lisbon are close to the same latitude; Moscow is farther north.

**11.** a) $y = 3 \cos(9000t)° + 8$
b) maximum equivalent stress
c) 6.64 MPa

**12.** Find the amplitude. Whatever the amplitude is, $a$ in the equation $y = a\cos(k(x - d)) + c$ will be equal to it. Find the period. Whatever the period is, $k$ in the equation $y = a\cos(k(x - d)) + c$ will be equal to 360 divided by it. Find the equation of the axis. Whatever the equation of the axis is, $c$ in the equation $y = a\cos(k(x - d)) + c$ will be equal to it. Find the phase shift. Whatever the phase shift is, $d$ in the equation $y = a\cos(k(x - d)) + c$ will be equal to it. Determine if the function is reflected in its axis. If it is, the sign of $a$ will be negative; otherwise, it will be positive. Determine if the function is reflected in the $y$-axis. If it is, the sign of $k$ will be negative; otherwise, it will be positive.

**13.** $y = -30 \cos(1.909\ 859x)° + 30$; 59.8 cm

**14.** $h = 7 \cos(22.74t)° + 8$, $t$ in seconds, $h$ in metres

## Lesson 6.7, pp. 398–401

**1.** a) $d = 1.5$ m, distance between tail lights and the curb if the trailer isn't swinging back and forth
b) amplitude: 0.5 m, distance the trailer swings to the left and right
c) period: 2 s, the time it takes for the trailer to swing back and forth
d) $d = 0.5 \sin(180t)° + 1.5$; $\{d \in \mathbf{R} \mid 1 \le d \le 2\}$
e) range is the distance the trailer swings back and forth; domain is time
f) 1.2 m

**2.** Answers may vary. For example,
a) $h = 10$ m, axle height
b) amplitude: 7 m, length of blade
c) period: 20 s, time in seconds to complete revolution
d) domain: $\{t \in \mathbf{R} \mid 0 \le t \le 140\}$;
range: $\{h \in \mathbf{R} \mid 3 \le h \le 17\}$
e) $h = -7 \cos(18x)° + 10$
f) period would be larger

**3.** Answers may vary. For example, $d = 4 \cos(90(t - 1))° + 8$

**4.** a) same period (24), same horizontal translation (12), different amplitude (2.5 and 10), different equations of the axis ($T = 17.5$ and $T = -20$). The top one is probably the interior temperature (higher, with less fluctuation).
b) domain (for both): $\{t \in \mathbf{R} \mid 0 \le t \le 48\}$;
range (top): $\{T \in \mathbf{R} \mid 15 \le T \le 20\}$;
range: (bottom): $\{T \in \mathbf{R} \mid -30 \le T \le -10\}$
c) Answers may vary. For example,
blue: $T = 2.5 \cos(15(h - 12))° + 17.5$;
red: $T = 10 \cos(15(h - 12))° - 20$

**5.** a) Answers may vary. For example, $d = 30 \cos[18(t - 12)]°$
b) Answers may vary. For example, $d = 9 \cos[18(t - 12)]°$

**6.** a) Answers may vary. For example, $d = -0.3 \cos(144t)° + 1.8$
b) amplitude: 0.3, height of crest relative to normal water level
c) 2 m
d) 16
e) Answers may vary. For example, $d = -0.3 \cos(120t)° + 1.8$

**7.** Answers may vary. For example, $C = 4.5 \cos(21\ 600t)°$

**8.** a) Answers may vary. For example, $h = 8 \cos(450(t - 0.2))° + 12$
b) domain: $\{\text{t} \in \mathbf{R}\}$; range: $\{h \in \mathbf{R} \mid 4 \le h \le 20\}$
c) $h = 12$cm, resting position of the spring
d) 6.3 cm

**9.** a) Answers may vary. For example, $h = -30 \cos[(1.43)x]° + 40$
b) domain: $\{d \in \mathbf{R} \mid 0 \le d \le 400\pi\}$;
range: $\{h \in \mathbf{R} \mid 10 \le h \le 70\}$
c) 69.7 cm

**10.** The periods are the same. The rabbit population has a higher average value and amplitude. The fox population increases when the rabbit population is above average and decreases when the rabbit population is below average.

**11.** the period, amplitude, location of the axis, and horizontal shift

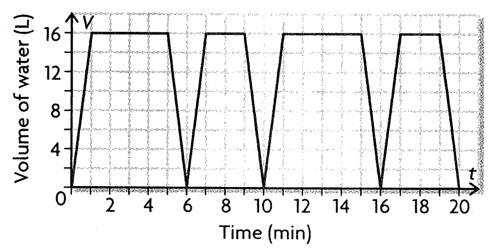
**12.** Answers may vary. For example, assuming the paint drop started at the lowest point $h = -6\cos(0.5x)° + 13$

**13.** Answers may vary. For example,
a) $f(x) = -3 \cos x - 1$ b) $-3.8$ c) (i) d) (iv)

**14.** 0°, 180°, 360°

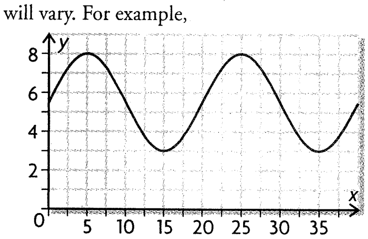
## Chapter Review, pp. 404–405

**1.** a)




b) yes
c) period: 10 min, how long it takes for the dishwasher to complete one cycle
d) $y = 8$ L
e) 8 L
f) $\{V \in \mathbf{R} \mid 0 \le V \le 16\}$

**2.** Answers will vary. For example,




**3.** Answers will vary. For example,