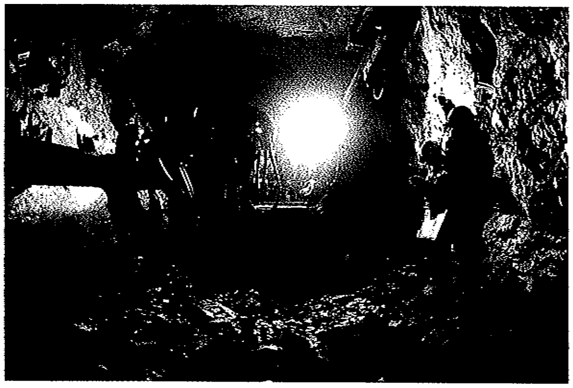
1.2 Function Notation

YOU WILL NEED
- graphing calculator

GOAL

Use function notation to represent linear and quadratic functions.

LEARN ABOUT the Math



The deepest mine in the world, East Rand mine in South Africa, reaches 3585 m into Earth's crust. Another South African mine, Western Deep, is being deepened to 4100 m. Suppose the temperature at the top of the mine shaft is 11 °C and that it increases at a rate of 0.015 °C/m as you descend.

? What is the temperature at the bottom of each mine?

EXAMPLE 1 Representing a situation with a function and using it to solve a problem

a) Represent the temperature in a mine shaft with a function. Explain why your representation is a function, and write it in **function notation.**

b) Use your function to determine the temperature at the bottom of East Rand and Western Deep mines.

function notation
notation, such as $f(x)$, used to represent the value of the dependent variable—the output—for a given value of the independent variable, x—the input

Communication | *Tip*

The notations y and $f(x)$ are interchangeable in the equation or graph of a function, so y is equal to $f(x)$. The notation $f(x)$ is read "f at x" or "f of x." The symbols $f(x)$, $g(x)$, and $h(x)$ are often used to name the outputs of functions, but other letters are also used, such as $v(t)$ for velocity as a function of time.

Lucy's Solution: Using an Equation

a) An equation for temperature is $T = 11 + 0.015d$, where T represents the temperature in degrees Celsius at a depth of d metres.

I wrote a linear equation for the problem.

I used the fact that T starts at 11 °C and increases at a steady rate of 0.015 °C/m.

The equation represents a function. Temperature is a function of depth.

Since this equation represents a linear relationship between temperature and depth, it is a function.

In function notation, $T(d) = 11 + 0.015d$.

I wrote the equation again. $T(d)$ makes it clearer that T is a function of d.

b) $T(3585) = 11 + 0.015(3585)$
$= 11 + 53.775$
$= 64.775$

I found the temperature at the bottom of East Rand mine by calculating the temperature at a depth of 3585 m. I substituted 3585 for *d* in the equation.

$T(4100) = 11 + 0.015(4100)$
$= 11 + 61.5$
$= 72.5$

For the new mine, I wanted the temperature when $d = 4100$, so I calculated $T(4100)$.

The temperatures at the bottom of East Rand mine and Western Deep mine are about 65 °C and 73 °C, respectively.

Stuart's Solution: Using a Graph

a) $T(d) = 11 + 0.015d$

This is a function because it is a linear relationship.

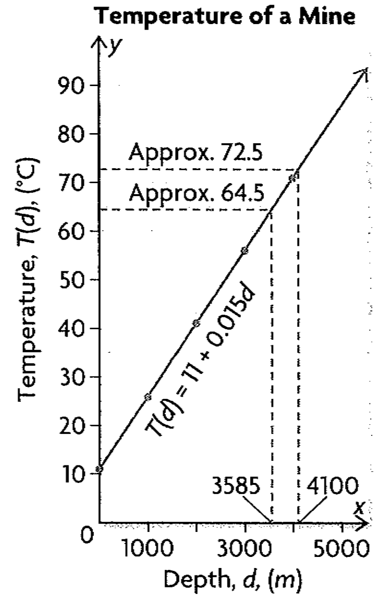
I wrote an equation to show how the temperature changes as you go down the mine.
I knew that the relationship was linear because the temperature increases at a steady rate. I used *d* for depth and called the function $T(d)$ for temperature.

b)

d (m)	*T*(*d*)(°C)
0	$T(0) = 11 + 0.015(0) = 11$
1000	26
2000	41
3000	56
4000	71

I made a table of values for the function.

I substituted the *d*-values into the function equation to get the $T(d)$-values.



I plotted the points (0, 11), (1000, 26), (2000, 41), (3000, 56), and (4000, 71). Then I joined them with a straight line.

East Rand mine is 3585 m deep. The temperature at the bottom is $T(3585)$.

I interpolated to read $T(3585)$ from the graph. It was approximately 65.

The other mine is 4100 m deep.

By extrapolating, I found that $T(4100)$ was about 73.

The temperature at the bottom of East Rand mine is about 65 °C. The temperature at the bottom of Western Deep mine is about 73 °C.

Tech | *Support*

For help using a graphing calculator to graph and evaluate functions, see Technical Appendix, B-2 and B-3.

Eli's Solution: Using a Graphing Calculator

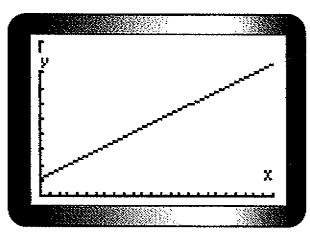
a) Let $T(d)$ represent the temperature in degrees Celsius at a depth of d metres.

I used function notation to write the equation.

$$T(d) = 11 + 0.015d$$

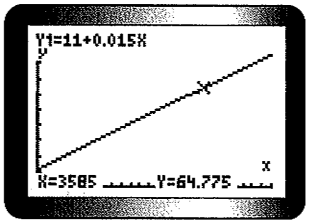
Temperature increases at a steady rate, so it is a function of depth.

b)

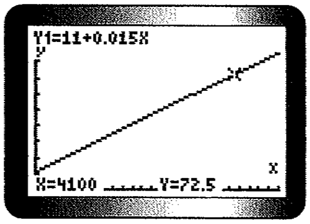


I graphed the function by entering Y1 = 11 + 0.015X into the equation editor.

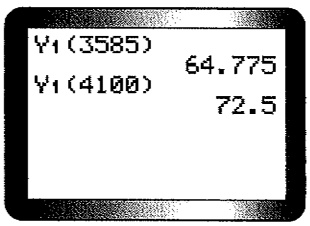
I used WINDOW settings of $0 \leq X \leq 5000$, Xscl 200, and $0 \leq Y \leq 100$, Yscl 10.



I used the value operation to find the temperature at the bottom of East Rand mine. This told me that $T(3585) = 64.775$.



Then I used the value operation again to find the temperature at the bottom of the other mine. I found that $T(4100) = 72.5$.



As a check, I called up the function on my calculator home screen, using **VARS** and function notation to display both answers.

The temperature at the bottom of the East Rand mine is about 65 °C. The temperature at the bottom of Western Deep mine is about 73 °C.

Reflecting

A. How did Lucy, Stuart, and Eli know that the relationship between temperature and depth is a function?

B. How did Lucy use the function equation to determine the two temperatures?

C. What does $T(3585)$ mean? How did Stuart use the graph to determine the value of $T(3585)$?

APPLY the Math

EXAMPLE 2 Representing a situation with a function model

A family played a game to decide who got to eat the last piece of pizza. Each person had to think of a number, double it, and subtract the result from 12. Finally, they each multiplied the resulting difference by the number they first thought of. The person with the highest final number won the pizza slice.

a) Use function notation to express the final answer in terms of the original number.

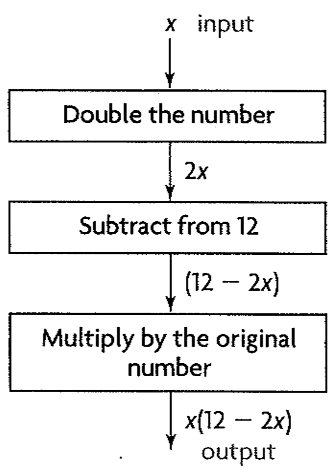
b) The original numbers chosen by the family members are shown. Who won the pizza slice?

c) What would be the best number to choose? Why?

Tim	5
Rhea	−2
Sara	7
Andy	10

Barbara's Solution

a)



I used a flow chart to show what happens to the original number in the game.

I chose x to be the original number, or input.

$$f(x) = x(12 - 2x)$$
$$= 12x - 2x^2$$

The expression for the final answer is quadratic, so the final result must be a function of the original number.

I chose $f(x)$ as the name for the final answer, or output.

b) Tim: $f(5) = 12(5) - 2(5)^2$

$= 60 - 2(25)$

$= 60 - 50$

$= 10$

I found the values of $f(5)$, $f(-2)$, $f(7)$, and $f(10)$ to see who had the highest answer.

Tim's answer was 10.

Rhea: $f(-2) = 12(-2) - 2(-2)^2$

$= -24 - 2(4)$

$= -24 - 8$

$= -32$

Rhea's answer was -32.

Sara: $f(7) = 12(7) - 2(7)^2$

$= 84 - 2(49)$

$= 84 - 98 = -14$

Sara's answer was -14.

Andy: $f(10) = 12(10) - 2(10)^2$

$= 120 - 2(100)$

$= 120 - 200 = -80$

Andy's answer was -80.

Tim won the pizza slice.

Tim's answer was the highest.

c) $f(x) = 12x - 2x^2$

I recognized that the equation was quadratic and that its graph would be a parabola that opened down, since the coefficient of x^2 was negative.

This meant that this quadratic function had a maximum value at its vertex.

$f(x) = -2x(x - 6)$

The x-intercepts are $x = 0$ and $x = 6$.

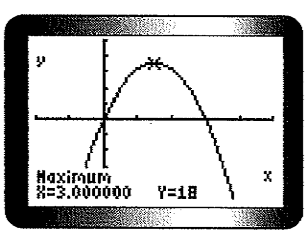
Vertex: $x = (0 + 6) \div 2$

$x = 3$

The best number to choose is 3.

I put the equation back in factored form by dividing out the common factor, $-2x$.

I remembered that the x-coordinate of the vertex is halfway between the two x-intercepts.

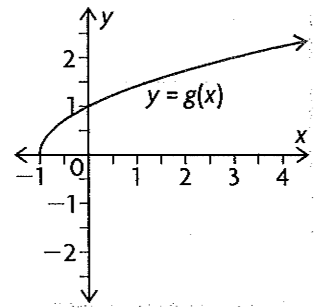


I checked my answer by graphing.

EXAMPLE 3 Connecting function notation to a graph

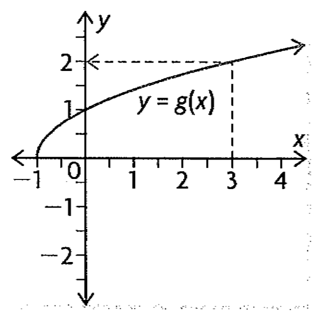
For the function shown in the graph, determine each value.

a) $g(3)$
b) $g(-1)$
c) x if $g(x) = 1$
d) the domain and range of $g(x)$



Ernesto's Solution

a)



I looked at the graph to find the y-coordinate when $x = 3$.

I drew a line up to the graph from the x-axis at $x = 3$ and then a line across from that point of intersection to the y-axis.

When $x = 3, y = 2$.

$g(3) = 2$

The y-value was 2, so, in function notation, $g(3) = 2$.

b) When $x = -1, y = 0$.

$g(-1) = 0$

I saw that $y = 0$ when $x = -1$, so -1 is the x-intercept and $g(-1) = 0$.

c) $g(x) = 1$ when $x = 0$

I saw that the graph crosses the y-axis at $y = 1$. The x-value is 0 at this point.

d) The graph begins at the point $(-1, 0)$ and continues upward. The graph exists only for $x \geq -1$ and $y \geq 0$.

The domain is all real numbers greater than or equal to -1.

The range is all real numbers greater than or equal to 0.

I saw that there was no graph to the left of the point $(-1, 0)$ or below that point.

So the only possible x-values are $x \geq -1$, and the only possible y-values are $y \geq 0$.

EXAMPLE 4 Using algebraic expressions in functions

Consider the functions $f(x) = x^2 - 3x$ and $g(x) = 1 - 2x$.

a) Show that $f(2) > g(2)$, and explain what that means about their graphs.

b) Determine $g(3b)$.

c) Determine $f(c + 2) - g(c + 2)$.

Jamilla's Solution

a) $f(x) = x^2 - 3x$

$f(2) = (2)^2 - 3(2)$

$= 4 - 6$

$= -2$

$g(x) = 1 - 2x$

$g(2) = 1 - 2(2)$

$= 1 - 4$

$= -3$

I substituted 2 for x in both functions.

$-2 > -3$, so $f(2) > g(2)$

That means that the point on the graph of $f(x)$ is above the point on the graph of $g(x)$ when $x = 2$.

b) $g(3b) = 1 - 2(3b)$

$= 1 - 6b$

I substituted $3b$ for x.

I simplified the equation.

c) $f(c + 2) - g(c + 2) = [(c + 2)^2 - 3(c + 2)] - [1 - 2(c + 2)]$

$= [(c^2 + 4c + 4 - 3c - 6)] - [1 - 2c - 4]$

$= [c^2 + c - 2] - [-3 - 2c]$

$= c^2 + c - 2 + 3 + 2c$

$= c^2 + 3c + 1$

I substituted $c + 2$ for x in both functions.

I used square brackets to keep the functions separate until I had simplified each one.

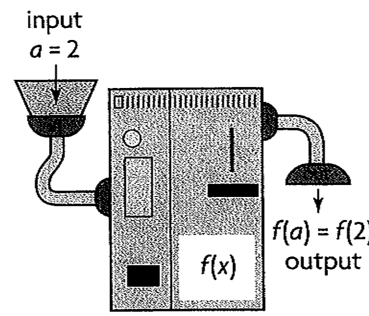
In Summary

Key Idea

- Symbols such as $f(x)$ are called function notation, which is used to represent the value of the dependent variable y for a given value of the independent variable x. For this reason, y and $f(x)$ are interchangeable in the equation of a function, so $y = f(x)$.

Need to Know

- $f(x)$ is read "f at x" or "f of x."
- $f(a)$ represents the value or output of the function when the input is $x = a$. The output depends on the equation of the function. To evaluate $f(a)$, substitute a for x in the equation for $f(x)$.
- $f(a)$ is the y-coordinate of the point on the graph of f with x-coordinate a. For example, if $f(x)$ takes the value 3 at $x = 2$, then $f(2) = 3$ and the point $(2, 3)$ lies on the graph of f.

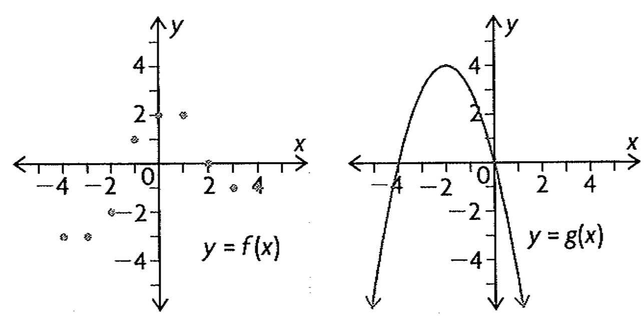


CHECK Your Understanding

1. Evaluate, where $f(x) = 2 - 3x$.

a) $f(2)$ **b)** $f(0)$ **c)** $f(-4)$ **d)** $f\left(\frac{1}{2}\right)$ **e)** $f(a)$ **f)** $f(3b)$

2. The graphs of $y = f(x)$ and $y = g(x)$ are shown.



Using the graphs, evaluate

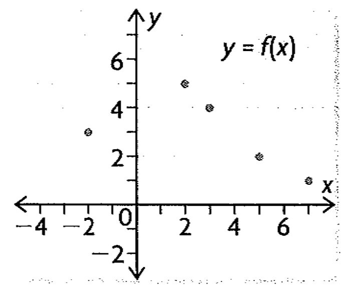
a) $f(1)$ **b)** $g(-2)$ **c)** $f(4) - g(-2)$ **d)** x when $f(x) = -3$

3. Milk is leaking from a carton at a rate of 3 mL/min. There is 1.2 L of milk in the carton at 11:00 a.m.
   a) Use function notation to write an equation for this situation.
   b) How much will be left in the carton at 1:00 p.m.?
   c) At what time will 450 mL of milk be left in the carton?

PRACTISING

4. Evaluate $f(-1), f(3)$, and $f(1.5)$ for
   a) $f(x) = (x - 2)^2 - 1$ b) $f(x) = 2 + 3x - 4x^2$

5. For $f(x) = \frac{1}{2x}$, determine
   a) $f(-3)$ b) $f(0)$ c) $f(1) - f(3)$ d) $f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)$

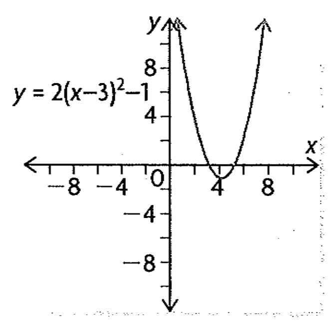


6. The graph of $y = f(x)$ is shown at the right.
   a) State the domain and range of f.
   b) Evaluate.
      i) $f(3)$
      ii) $f(5)$
      iii) $f(5 - 3)$
      iv) $f(5) - f(3)$

7. For $h(x) = 2x - 5$, determine
   a) $h(a)$
   b) $h(b + 1)$
   c) $h(3c - 1)$
   d) $h(2 - 5x)$

8. Consider the function $g(t) = 3t + 5$.
   a) Create a table of values and graph the function.
   b) Determine each value.
      i) $g(0)$
      ii) $g(3)$
      iii) $g(1) - g(0)$
      iv) $g(2) - g(1)$
      v) $g(1001) - g(1000)$
      vi) $g(a + 1) - g(a)$

9. Consider the function $f(s) = s^2 - 6s + 9$.
   a) Create a table of values for the function.
   b) Determine each value.
      i) $f(0)$
      ii) $f(1)$
      iii) $f(2)$
      iv) $f(3)$
      v) $[f(2) - f(1)] - [f(1) - f(0)]$
      vi) $[f(3) - f(2)] - [f(2) - f(1)]$
   c) In part (b), what do you notice about the answers to parts (v) and (vi)? Explain why this happens.

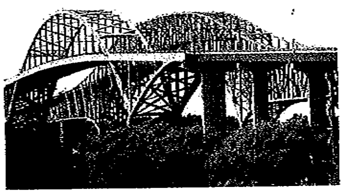


10. The graph at the right shows $f(x) = 2(x - 3)^2 - 1$.
   K a) Evaluate $f(-2)$.
   b) What does $f(-2)$ represent on the graph of f?
   c) State the domain and range of the relation.
   d) How do you know that f is a function from its graph?

11. For $g(x) = 4 - 5x$, determine the input for x when the output of $g(x)$ is
   a) -6 b) 2 c) 0 d) $\frac{3}{5}$

12. A company rents cars for $50 per day plus $0.15/km.
   a) Express the daily rental cost as a function of the number of kilometres travelled.
   b) Determine the rental cost if you drive 472 km in one day.
   c) Determine how far you can drive in a day for $80.

13. **A** As a mental arithmetic exercise, a teacher asked her students to think of a number, triple it, and subtract the resulting number from 24. Finally, they were asked to multiply the resulting difference by the number they first thought of.
   a) Use function notation to express the final answer in terms of the original number.
   b) Determine the result of choosing numbers 3, -5, and 10.
   c) Determine the maximum result possible.



14. **T** The second span of the Bluewater Bridge in Sarnia, Ontario, is supported by two parabolic arches. Each arch is set in concrete foundations that are on opposite sides of the St. Clair River. The feet of the arches are 281 m apart. The top of each arch rises 71 m above the river. Write a function to model the arch.

15. a) Graph the function $f(x) = 3(x - 1)^2 - 4$.
   b) What does $f(-1)$ represent on the graph? Indicate on the graph how you would find $f(-1)$.
   c) Use the equation to determine
      i) $f(2) - f(1)$ ii) $2f(3) - 7$ iii) $f(1 - x)$

16. Let $f(x) = x^2 + 2x - 15$. Determine the values of x for which
   a) $f(x) = 0$ b) $f(x) = -12$ c) $f(x) = -16$

17. Let $f(x) = 3x + 1$ and $g(x) = 2 - x$. Determine values for a such that
   a) $f(a) = g(a)$ b) $f(a^2) = g(2a)$

18. **C** Explain, with examples, what function notation is and how it relates to the graph of a function. Include a discussion of the advantages of using function notation.

Extending

19. The highest and lowest marks awarded on an examination were 285 and 75. All the marks must be reduced so that the highest and lowest marks become 200 and 60.
   a) Determine a linear function that will convert 285 to 200 and 75 to 60.
   b) Use the function to determine the new marks that correspond to original marks of 95, 175, 215, and 255.

20. A function $f(x)$ has these properties:
   - The domain of f is the set of natural numbers.
   - $f(1) = 1$
   - $f(x + 1) = f(x) + 3x(x + 1) + 1$
   a) Determine $f(2), f(3), f(4), f(5)$, and $f(6)$.
   b) Describe the function.