8.4 Applying the Cosine Law

YOU WILL NEED

- ruler

GOAL

Use the cosine law to calculate unknown measures of sides and angles in acute triangles.

LEARN ABOUT the Math

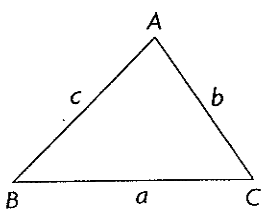
In Lesson 8.3, you discovered the cosine law for acute triangles. Can you be sure that the cosine law is true for every acute triangle?

? **How can you show that the cosine law is true for all acute triangles?**

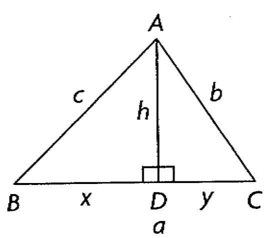
EXAMPLE 1 Proving the cosine law for acute triangles

Show that the cosine law is true for all acute triangles.

Heather's Solution



I started by drawing an acute triangle *ABC*.



The cosine law is just an extension of the Pythagorean theorem. I thought that I might be able to relate the angles and sides in $\triangle ABC$ if I could create right triangles. I drew a line segment from *A* to *D* so that it was perpendicular to *BC*. I labelled this line segment *h*. I used *x* for *BD* and *y* for *DC*.

$c^2 = h^2 + x^2$, so $h^2 = c^2 - x^2$

$b^2 = h^2 + y^2$, so $h^2 = b^2 - y^2$

I wrote the Pythagorean theorem for each triangle to determine two different expressions for h^2.

$c^2 - x^2 = b^2 - y^2$

Then I set the two expressions equal.

$x = a - y$, so

$$c^2 - (a - y)^2 = b^2 - y^2$$
$$c^2 = (a - y)^2 + b^2 - y^2$$
$$c^2 = a^2 - 2ay + y^2 + b^2 - y^2$$
$$c^2 = a^2 + b^2 - 2ay$$

I didn't want to include both *x* and *y* in the same equation. I only wanted one of them. So, I substituted $x = a - y$ for *x*. I simplified the equation by expanding and collecting like terms.

NEL

$\cos C = \frac{y}{b}$, so

$b \cos C = y$

The cosine law for this triangle had to include side lengths a, b, and c, as well as one angle. My equation $c^2 = a^2 + b^2 - 2ay$ included the three side lengths, but it also included y, which I didn't know. My equation did not involve any angles.

I had to write y in terms of one of the angles in the triangle. Since y is adjacent to $\angle C$ in $\triangle ADC$, I decided to write y in terms of the cosine of $\angle C$.

$c^2 = a^2 + b^2 - 2ay$

$c^2 = a^2 + b^2 - 2\,ab \cos C$

I substituted the expression $b \cos C$ for y into my equation.

Reflecting

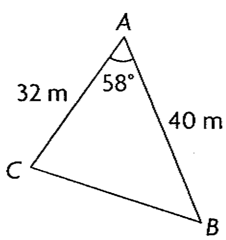
A. Why did it make sense for Heather to divide the acute triangle into two right triangles?

B. Suppose that Heather had substituted $a - x$ for y instead of $a - y$ for x. Would her result have been the same? How do you know?

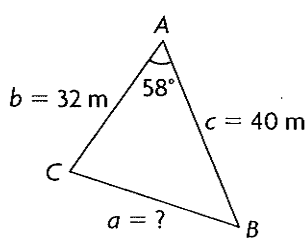
APPLY the Math

EXAMPLE 2 Selecting a cosine law strategy to calculate the length of a side

Determine the length of CB.



Justin's Solution



I copied the triangle and named the sides using lower-case letters. Then I identified the measure that I had to determine. Since the triangle did not contain a right angle, I couldn't use primary trigonometric ratios. I couldn't use the sine law either, because I didn't know a side length and the measure of its opposite angle.

$a^2 = b^2 + c^2 - 2\,bc \cos A$

$a^2 = 32^2 + 40^2 - 2(32)(40)\cos 58°$

I knew two sides (b and c) and the angle between these sides ($\angle A$). I had to determine side a, which is opposite $\angle A$. The cosine law relates these four measurements, so I substituted the values I knew into the cosine law.

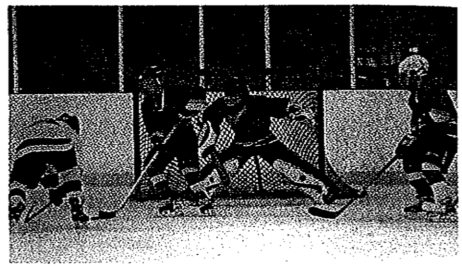
$$a^2 = 1024 + 1600 - 2560 \cos 58°$$
$$a^2 = 2624 - 2560 \cos 58°$$
$$a^2 = 1267.41$$
$$a = \sqrt{1267.41}$$
$$a \doteq 35.6$$

I evaluated the right side. Then I calculated the square root.

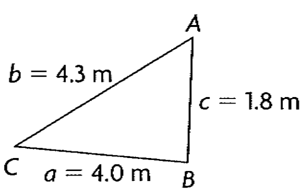
CB is about 36 m.

EXAMPLE 3 Selecting a cosine law strategy to calculate the measure of an angle

The posts of a hockey net are 1.8 m apart. A player tries to score a goal by shooting the puck along the ice from a point that is 4.3 m from one goalpost and 4.0 m from the other goalpost. Determine the measure of the angle that the puck makes with both goalposts.



Darcy's Solution



I drew a diagram to represent the situation. I couldn't assume that this triangle contained a right angle, so I couldn't use primary trigonometric ratios. I didn't know the measure of an angle and a side length opposite the angle, so I couldn't use the sine law.

$$c^2 = a^2 + b^2 - 2\,ab \cos C$$
$$1.8^2 = 4.0^2 + 4.3^2 - 2(4.0)(4.3) \cos C$$

I had to determine the measure of $\angle C$. The cosine law relates the three sides of a triangle to an angle in the triangle. I substituted the measures of the sides in this triangle into the cosine law.

$$3.24 = 16.00 + 18.49 - 34.40 \cos C$$
$$3.24 - 16.00 - 18.49 = -34.40 \cos C$$
$$-31.25 = -34.40 \cos C$$
$$\frac{-31.25}{-34.40} = \cos C$$
$$0.9084 \doteq \cos C$$

I simplified and then solved for cos *C*.

$$\cos^{-1}(0.9084) = \angle C$$
$$24.7° \doteq \angle C$$

I used the inverse cosine to calculate $\angle C$.

The puck makes an angle of about 25° with the goalposts.

In Summary

Key Idea

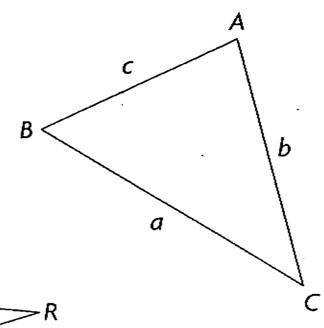
- The cosine law can be used to determine an unknown side length or angle measure in an acute triangle.

Need to Know

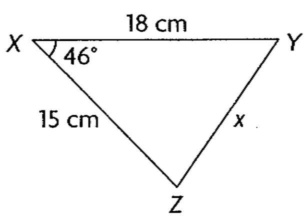
- You can use the cosine law to solve a problem that can be modelled by an acute triangle if you can determine the measurements of
  - two sides and the angle between them
  - all three sides
- An acute triangle can be divided into smaller right triangles by drawing a perpendicular line from a vertex to the opposite side. The proof of the cosine law involves applying the Pythagorean theorem and cosine ratio to these right triangles.

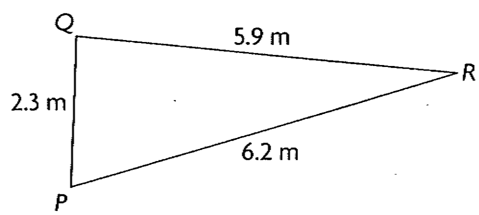
CHECK Your Understanding

1. Suppose that you are given each set of data for $\triangle ABC$ at the right. Can you use the cosine law to determine c? Explain.
   a) $a = 5$ cm, $\angle A = 52°$, $\angle C = 43°$
   b) $a = 5$ cm, $b = 7$ cm, $\angle C = 43°$



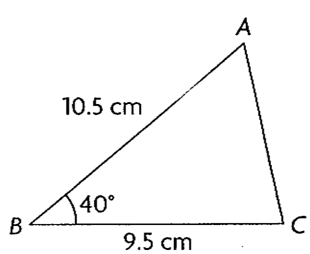
2. a) Determine the length of side x. b) Determine the measure of $\angle P$.



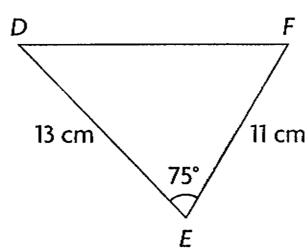


PRACTISING

3. Determine each unknown side length.
   a)

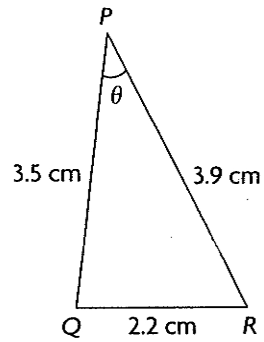


   b)

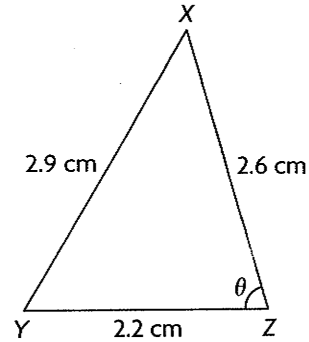


4. Determine the measure of each indicated angle to the nearest degree.

a)



b)



5. Solve each triangle.

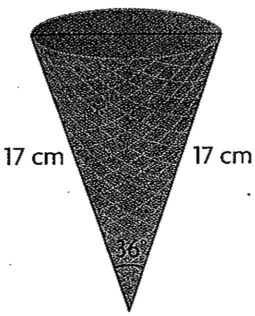
K a) In $\triangle DEF$, $d = 5.0$ cm, $e = 6.5$ cm, and $\angle F = 65°$.

b) In $\triangle PQR$, $p = 6.4$ m, $q = 9.0$ m, and $\angle R = 80°$.

c) In $\triangle LMN$, $l = 5.5$ cm, $m = 4.6$ cm, and $n = 3.3$ cm.

d) In $\triangle XYZ$, $x = 5.2$ mm, $y = 4.0$ mm, and $z = 4.5$ cm.

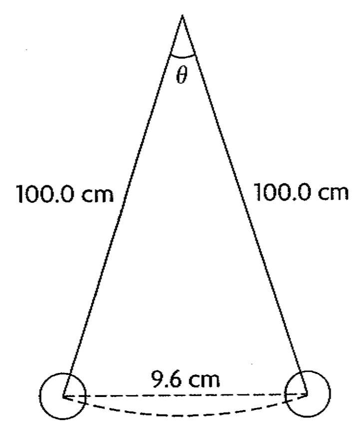
6. Determine the perimeter of $\triangle SRT$, if $\angle S = 60°$, $r = 15$ cm, and $t = 20$ cm.



7. An ice cream company is designing waffle cones to use for serving frozen yogurt. The cross-section of the design has a bottom angle of 36°. The sides of the cone are 17 cm long. Determine the diameter of the top of the cone.

8. A parallelogram has sides that are 8 cm and 15 cm long. One of the **C** angles in the parallelogram measures 70°. Explain how you could calculate the length of the shortest diagonal.

9. The pendulum of a grandfather clock is **A** 100.0 cm long. When the pendulum swings from one side to the other side, the horizontal distance it travels is 9.6 cm, as in the diagram at the right. Determine the angle through which the pendulum swings. Round your answer to the nearest tenth of a degree.

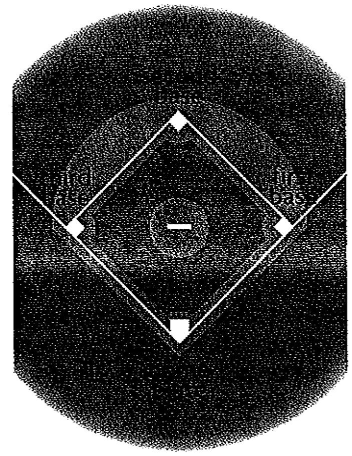


10. a) A clock has a minute hand that is 20 cm long and an hour hand that is 12 cm long. Calculate the distance between the tips of the hands at

i) 2:00 ii) 10:00

b) Discuss your results for part a).

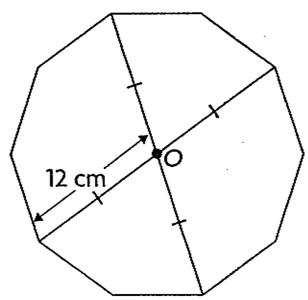
11. The bases in a baseball diamond are 90 ft apart. A player picks up a ground ball 11 ft from third base, along the line from second base to third base. Determine the angle that is formed between first base, the player's present position, and home plate.



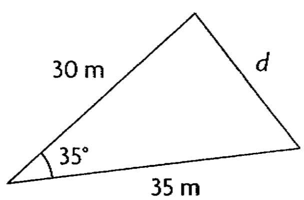
12. Sally makes stained glass windows. Each piece of glass is surrounded by lead edging. Sally claims that she can create an acute triangle in part of a window using pieces of lead that are 15 cm, 36 cm, and 60 cm. Is she correct? Justify your decision.

13. Two drivers leave home at the same time and travel on straight roads that diverge by 70°. One driver travels at an average speed of 83.0 km/h. The other driver travels at an average speed of 95.0 km/h. How far apart will the two drivers be after 45 min?

14. **T** The distance from the centre, O, of a regular decagon to each vertex is 12 cm. Calculate the area of the decagon.



15. Use the triangle at the right to create a problem that involves side lengths and interior angles. Then describe how to determine the length of side d.



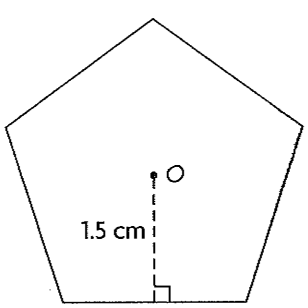
History Connection

The first baseball game recorded in Canada was played in Beachville, Ontario, on June 4, 1838.

Extending

16. An airplane is flying from Montréal to Vancouver. The wind is blowing from the west at 60 km/h. The airplane flies at an airspeed of 750 km/h and must stay on a heading of 65° west of north.
    a) What heading should the pilot take to compensate for the wind?
    b) What is the speed of the airplane relative to the ground?

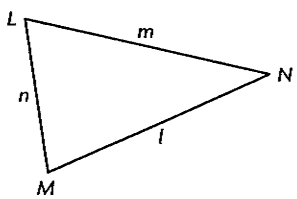
17. Calculate the perimeter and area of this regular pentagon. O is the centre of this pentagon.



4. a) $\theta \doteq 43°$, $x \doteq 5.9$ cm
   b) $\theta = 62°$, $x \doteq 10.6$ cm, $y \doteq 9.7$ cm
5. $\angle C = 60°$, $b \doteq 12.2$ cm, $c \doteq 13.8$ cm
6. $\angle X$ or $\angle Z$
7. a) the right tower
   b) about 3.1 km
8. about 300 m
9. a) about 84 cm
   b) about 82 cm

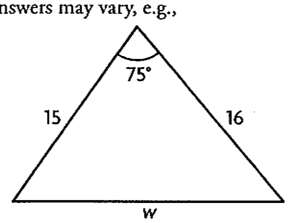
Lesson 8.3, page 438

1. a)



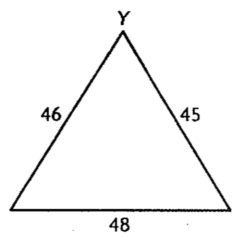
   b) $l^2 = m^2 + n^2 - 2\,mn\cos L$, $m^2 = l^2 + n^2 - 2\,ln\cos M$, $n^2 = l^2 + m^2 - 2\,lm\cos N$
2. Answers may vary, e.g.,

a)



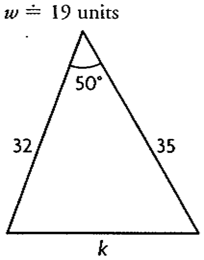
$w \doteq 19$ units

c)



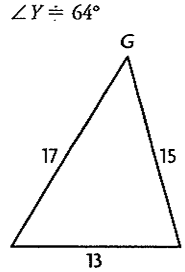
$\angle Y \doteq 64°$

b)



$k \doteq 28$ units

d)



$\angle G \doteq 47°$

3. Answers may vary, e.g.,
   a) If an angle measure needs to be determined, the cosine ratio can be determined quickly and the angle measure can be determined using $\cos^{-1}$.
   b) $\dfrac{q^2 + r^2 - p^2}{2\,qr} = \cos P$
   c) $\dfrac{p^2 + r^2 - q^2}{2\,pr} = \cos Q$
4. a) the angle opposite the side with the unknown length and the other two sides
   b) all three side lengths
5. The square of a side length equals the sum of the squares of the other two side lengths minus twice the product of the other two side lengths and the cosine of the angle opposite the first side length.

Lesson 8.4, page 443

1. a) No. Another side length, b, is required.
   b) Yes. The lengths of two sides and the measure of the angle between them are given.
2. a) about 13.2 cm
   b) about 72°
3. a) about 6.9 cm
   b) about 14.7 cm
4. a) 34°
   b) 74°
5. a) $\angle D \doteq 46°$, $\angle E \doteq 69°$, $f \doteq 6.3$ cm
   b) $\angle P \doteq 39°$, $\angle Q \doteq 61°$, $r \doteq 10.1$ m
   c) $\angle L \doteq 87°$, $\angle M \doteq 57°$, $\angle N \doteq 37°$
   d) $\angle X \doteq 75°$, $\angle Y \doteq 48°$, $\angle Z \doteq 57°$
6. about 53 cm
7. about 11 cm
8. Use the cosine law: the diagonal, $d^2 = 8^2 + 15^2 - 2(8)(15)\cos 70°$.
9. 5.5°
10. a) i) about 17 cm   ii) about 17 cm
    b) Answers may vary, e.g., the lengths are equal because the triangles formed at 2:00 and at 10:00 are congruent triangles.
11. about 48°
12. No. The angle opposite the 60 cm side would have a negative cosine, which is impossible for an acute angle.
13. about 76.9 km
14. about 423 cm^2
15. Answers will vary, e.g., Problem: Joe and Marie swim away from each other at an angle of 35°. Joe swims at 6 m/s, and Marie swims at 7 m/s. How far apart are they after 5 s? Answer: Joe's distance after 5 s is 30 m, and Marie's distance after 5 s is 35 m. Use the cosine law to determine d, the distance they are apart.
    Problem: During a game of golf, Andrew's ball is 30 m from the hole and Brett's ball is 35 m from the hole. The angle between the two balls, when viewed from the hole, is 35°. How far apart are the two balls? Answer: $d^2 = 30^2 + 35^2 - 2(30)(35)\cos 35°$
16. a) about 67° west of north   b) about 805 km/h
17. perimeter: about 10.9 cm; area: about 8.2 cm^2

Lesson 8.5, page 449

1. a) sine law
   b) tangent ratio or sine law
   c) cosine law
2. a) about 84°   b) about 1.9 cm   c) about 40°
3. a) 64°   b) about 16 cm   c) about 52 cm
4. about 2.5 km
5. about 241 m
6. 8.9 m, 9.5 m
7. a) about 43 m   b) about 13 m
8. *Albacore*: about 61 km; *Bonito*: about 39 km
9. about 276 m
10. about 3.8 km
11. a) about 11 m
    b) about 19 m
12. about 59 cm
13. a) about 879 m
    b) about 40 s
14. a) about 157 km
    b) The airplane that is 100 km away will arrive first.
15. about 85°, about 95°, about 85°, about 95°