6.4 The Quadratic Formula

YOU WILL NEED

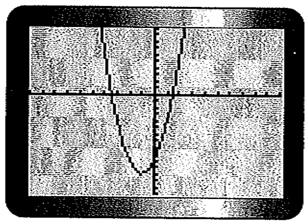
- graphing calculator

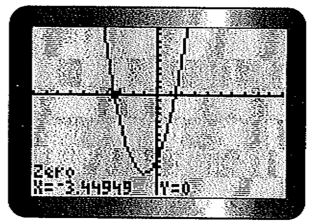
GOAL

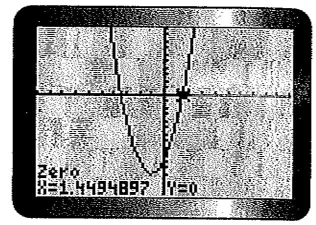
Understand the development of the quadratic formula, and use the quadratic formula to solve quadratic equations.

LEARN ABOUT the Math

Devlin says that he cannot solve the equation $2x^2 + 4x - 10 = 0$ by factoring because his graphing calculator shows him that the zeros of the relation $y = 2x^2 + 4x - 10$ are not integers.







He wonders if there is a way to solve quadratic equations that cannot be factored over the set of integers.

? How can quadratic equations be solved without factoring or using a graph?

EXAMPLE 1 **Selecting a strategy to solve a quadratic equation**

Solve $2x^2 + 4x - 10 = 0$.

Kyle's Solution: Solving a quadratic equation using the vertex form

$$2x^2 + 4x - 10 = 0$$

Since the equation contains an x^2 term as well as an x term, I knew that I couldn't isolate x like I do for linear equations. But the vertex form of a quadratic equation does contain a single x term. I wondered whether I could isolate x if I wrote the expression on the left in this form. I decided to complete the square.

$$2(x^2 + 2x) - 10 = 0$$
$$\frac{2}{2} = 1 \text{ and } 1^2 = 1$$
$$2(x^2 + 2x + 1 - 1) - 10 = 0$$

I factored 2 from the x^2 and x terms. I divided the coefficient of x by 2. Then I squared my result to determine what I needed to add and subtract to create a perfect square within the expression on the left side.

$$\begin{aligned} 2[(x^2 + 2x + 1) - 1] - 10 &= 0 \\ 2[(x + 1)^2 - 1] - 10 &= 0 \\ 2(x + 1)^2 - 2 - 10 &= 0 \\ 2(x + 1)^2 - 12 &= 0 \end{aligned}$$

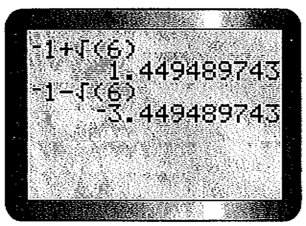
I grouped together the three terms that formed the perfect-square trinomial and then factored. Finally, I multiplied and combined the constants.

$$\begin{aligned} 2(x + 1)^2 &= 12 \\ \frac{2(x + 1)^2}{2} &= \frac{12}{2} \\ (x + 1)^2 &= 6 \\ \sqrt{(x + 1)^2} &= \pm\sqrt{6} \\ x + 1 &= \pm\sqrt{6} \\ x &= -1 \pm \sqrt{6} \end{aligned}$$

I isolated $(x + 1)^2$ using inverse operations. Since $(x + 1)$ is squared, I took the square roots of both sides. I remembered that there are two square roots for every number: a positive one and a negative one. Then I solved for x.

$x = -1 + \sqrt{6}$ and $x = -1 - \sqrt{6}$ are the exact solutions.

I got two answers. This makes sense because these roots are the x-intercepts of the graph of $y = 2x^2 + 4x - 10$.



I decided to compare the roots I calculated with the x-intercepts of Devlin's graph by writing these numbers as decimals. My roots and Devlin's x-intercepts were the same.

The roots of $2x^2 + 4x - 10 = 0$ are approximately $x = 1.4$ and $x = -3.4$.

Liz's Solution: Solving a quadratic equation by developing a formula

$$ax^2 + bx + c = 0$$

I thought I could solve for x if I could write the standard form of an equation in vertex form, since this form is the only one that has a single x term. I realized that I would have to work with letters instead of numbers, but I reasoned that the process would be the same.

$$a\left(x^2 + \frac{b}{a}x\right) + c = 0$$

I decided to complete the square, so I factored a from the x^2 and x terms.

$$\frac{b}{a} \div 2 = \frac{b}{a} \times \frac{1}{2} = \frac{b}{2a} \text{ and } \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

To determine what I needed to add and subtract to create a perfect square within the expression, I divided the coefficient of x by 2. Then I squared my result.

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c = 0$$

$$a\left[\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a^2}\right] + c = 0$$

$$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{ab^2}{4a^2} + c = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0$$

I added and subtracted $\frac{b^2}{4a^2}$ to the binomial inside the brackets. I grouped together the terms that formed the perfect-square trinomial and then factored. Finally, I multiplied by a and simplified.

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

To solve for x, I used inverse operations. I divided both sides by a. I took the square root of both sides. Since the right side represents a number, I had to determine both the positive and negative square roots.

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

I subtracted $\frac{b}{2a}$ from both sides and simplified the expression.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

I reasoned that my formula could be used for any quadratic equation. The $\pm$ in the numerator means that there could be two solutions: one when you add the square root of $b^2 - 4ac$ to $-b$ before dividing by $2a$, and another when you subtract.

$$2x^2 + 4x - 10 = 0$$

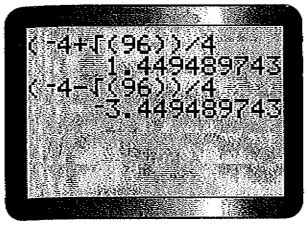
$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(-10)}}{2(2)}$$

$$x = \frac{-4 \pm \sqrt{16 + 80}}{4}$$

$$x = \frac{-4 \pm \sqrt{96}}{4}$$

To verify my formula, I checked that the roots were the same as Devlin's x-intercepts. I substituted $a = 2$, $b = 4$, and $c = -10$ into my formula.

The roots of $2x^2 + 4x - 10 = 0$ are $x = \frac{-4 + \sqrt{96}}{4}$ and $x = \frac{-4 - \sqrt{96}}{4}$.



I calculated my roots as decimals so that I could compare them with Devlin's x-intercepts. They were the same.

Reflecting

A. Why does it make sense that a quadratic equation may have two solutions?

B. How are Kyle's solution and Liz's solution the same? How are they different?

C. Why does it make sense that a, b, and c are part of the **quadratic formula**?

quadratic formula
a formula for determining the roots of a quadratic equation of the form $ax^2 + bx + c = 0$; the quadratic formula is written using the coefficients and the constant in the equation:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

APPLY the Math

EXAMPLE 2 Selecting a tool to verify the roots of a quadratic equation

Solve $5x^2 - 4x - 3 = 0$. Round your solutions to two decimal places. Verify your solutions using a graphing calculator.

Maddy's Solution

$$5x^2 - 4x - 3 = 0$$
$$a = 5, b = -4, c = -3$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

I noticed that the trinomial in this equation is not factorable over the set of integers, so I decided to use the quadratic formula. I identified the values of a, b, and c.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(5)(-3)}}{2(5)}$$

I substituted the values for a, b, and c and simplified.

$$x = \frac{4 \pm \sqrt{16 + 60}}{10}$$
$$x = \frac{4 \pm \sqrt{76}}{10}$$
$$x \doteq \frac{4 \pm 8.718}{10}$$

I calculated $\sqrt{76}$ and rounded to 3 decimal places.

$x \doteq \frac{4 - 8.718}{10}$ or $x \doteq \frac{4 + 8.718}{10}$

$x = -0.4718 \quad x = 1.2718$

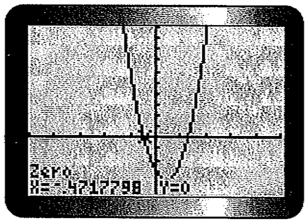
I knew that the ± in the formula meant that I would have two different solutions. I wrote the two solutions separately.

$x \doteq -0.47$ and $x \doteq 1.27$

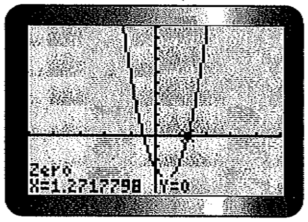
I rounded to two decimal places.

Tech | *Support*

For help using a TI-83/84 graphing calculator to determine the zeros of a relation, see Appendix B-8. If you are using a TI-*n*spire, see Appendix B-44.



I entered the relation $y = 5x^2 - 4x - 3$ into my graphing calculator and used the Zero operation to verify my solutions.



The zeros of the relation agreed with the roots I had calculated using the formula.

EXAMPLE 3 Reasoning about solving quadratic equations

Solve each equation. Round your solutions to two decimal places.

a) $2x^2 - 10 = 8$ **b)** $3x(5x - 4) + 2x = x^2 - 4(x - 3)$

Graham's Solution

a) $2x^2 - 10 = 8$

I noticed that this quadratic equation did not contain an x term. I reasoned that I could solve the equation if I isolated the x^2 term.

$$2x^2 = 8 + 10$$
$$2x^2 = 18$$

I added 10 to both sides.

$$\frac{2x^2}{2} = \frac{18}{2}$$

I divided both sides by 2.

$$x^2 = 9$$
$$\sqrt{x^2} = \pm\sqrt{9}$$
$$x = \pm 3$$

I took the square root of both sides.

$x = 3$ and $x = -3$

b) $3x(5x - 4) + 2x = x^2 - 4(x - 3)$ ← I simplified the equation by multiplying, using the distributive property.

$15x^2 - 12x + 2x = x^2 - 4x + 12$

$14x^2 - 6x - 12 = 0$ ← I rearranged the equation so that the right side was 0. I decided to use the quadratic formula since I didn't quickly see the factors.

$a = 14, b = -6, c = -12$ ← I identified the values of *a*, *b*, and *c*.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(14)(-12)}}{2(14)}$$

$$x = \frac{6 \pm \sqrt{708}}{28}$$

← I substituted the values of *a*, *b*, and *c* into the quadratic formula and evaluated. I rounded the solutions to two decimal places.

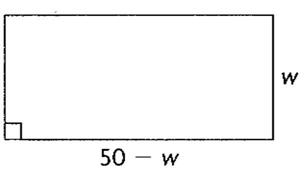
$x \doteq -0.74$ and $x \doteq 1.16$

EXAMPLE 4 Solving a problem using a quadratic model

A rectangular field is going to be completely enclosed by 100 m of fencing. Create a quadratic relation that shows how the area of the field will depend on its width. Then determine the dimensions of the field that will result in an area of 575 m^2. Round your answers to the nearest hundredth of a metre.

Bruce's Solution

Let the width of the field be w metres.



← I started with a diagram to help me organize my thinking. I decided to represent the width of the field by *w*.

Since the perimeter will be 100 m, the length will have to be $\frac{100 - 2w}{2} = 50 - w$.

$$\begin{aligned} A &= lw \\ &= w(50 - w) \\ &= 50w - w^2 \\ 575 &= 50w - w^2 \end{aligned}$$

← I wrote a quadratic relation for the area by multiplying the length and the width. Then I set the area equal to 575.

$0 = -w^2 + 50w - 575$ ← I rearranged the equation so that the left side was 0.

$a = -1, b = 50, c = -575$ ← I decided to use the quadratic formula, so I identified the values of *a*, *b*, and *c*.

$$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$w = \frac{-50 \pm \sqrt{50^2 - 4(-1)(-575)}}{2(-1)}$$

I substituted these values into the formula and evaluated. I rounded the solutions to two decimal places.

$$w = \frac{-50 \pm \sqrt{2500 - 2300}}{-2}$$

$$w = \frac{-50 \pm \sqrt{200}}{-2}$$

$w \doteq 17.93$ and $w \doteq 32.07$

The field is either 17.93 m or 32.07 m wide.

$50 - 17.93 = 32.07$

If $w = 17.93$ m, the field is 32.07 m long.

$50 - 32.07 = 17.93$

If $w = 32.07$ m, the field is 17.93 m long.

I used the two possible values of w to determine the values for the length. It made sense that the roots could be either the length or the width.

In Summary

Key Idea

- The roots of a quadratic equation of the form $ax^2 + bx + c = 0$ can be determined using the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Need to Know

- The quadratic formula was developed by completing the square to solve $ax^2 + bx + c = 0$.
- The quadratic formula provides a way to calculate the roots of a quadratic equation without graphing or factoring.
- The solutions to the equation $ax^2 + bx + c = 0$ correspond to the zeros, or x-intercepts, of the relation $y = ax^2 + bx + c$.
- Quadratic equations that do not contain an x term can be solved by isolating the x^2 term.
- Quadratic equations of the form $a(x - h)^2 + k = 0$ can be solved by isolating the x term.

CHECK Your Understanding

1. State the values of a, b, and c that you would substitute into the quadratic formula to solve each equation. Rearrange the equation, if necessary.
   a) $x^2 + 5x - 2 = 0$
   b) $4x^2 - 3 = 0$
   c) $x^2 + 6x = 0$
   d) $2x(x - 5) = x^2 + 1$

2. i) Solve each equation by factoring.
   ii) Solve each equation using the quadratic formula.
   iii) State which strategy you prefer for each equation, and explain why.
   a) $x^2 + 18x - 63 = 0$ b) $8x^2 - 10x - 3 = 0$

3. Solve each equation.
   a) $2x^2 = 50$ c) $3x^2 - 2 = 10$
   b) $x^2 - 1 = 0$ d) $x(x - 2) = 36 - 2x$

PRACTISING

4. Determine the roots of each equation. Round the roots to two decimal places, if necessary.
   a) $(x + 1)^2 - 16 = 0$ d) $4(x - 2)^2 - 5 = 0$
   b) $-2(x + 5)^2 + 2 = 0$ e) $-6(x + 3)^2 + 12 = 0$
   c) $-3(x - 7)^2 + 3 = 0$ f) $0.25(x - 4)^2 - 4 = 0$

5. Solve each equation using the quadratic formula.
   a) $6x^2 - x - 15 = 0$ d) $5x^2 - 11x = 0$
   b) $4x^2 - 20x + 25 = 0$ e) $x^2 + 9x + 20 = 0$
   c) $x^2 - 16 = 0$ f) $12x^2 - 40 = 17x$

6. Could you have solved the equations in question 5 using a different strategy? Explain.

7. **C** If you can solve a quadratic equation by factoring it over the set of integers, what would be true about the roots you could determine using the quadratic formula? Explain.

8. **K** Determine the roots of each equation. Round the roots to two decimal places.
   a) $x^2 - 4x - 1 = 0$ d) $2x^2 - x - 3 = 0$
   b) $5x^2 - 6x - 2 = 0$ e) $m^2 - 5m + 3 = 0$
   c) $3w^2 + 8w + 2 = 0$ f) $-3x^2 + 12x - 7 = 0$

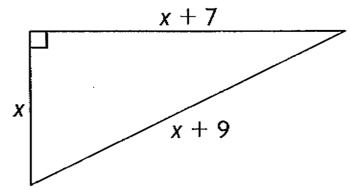
9. Solve each equation. Round your solutions to two decimal places.
   a) $2x^2 - 5x = 3(x + 4)$ d) $3x(x + 4) = (4x - 1)^2$
   b) $(x + 4)^2 = 2(x + 5)$ e) $(x - 2)(2x + 3) = x + 1$
   c) $x(x + 3) = 5 - x^2$ f) $(x - 3)^2 + 5 = 3(x + 1)$

10. Solve each equation. Round your solutions to two decimal places.
   a) $2x^2 + 5x - 14 = 0$ c) $3x(0.4x + 1) = 8.4$
   b) $3x^2 + 7.5x = 21$ d) $0.2x^2 = -0.5x + 1.4$

11. a) What do you notice about your solutions in question 10?
   b) How could you have predicted this before using the quadratic formula?

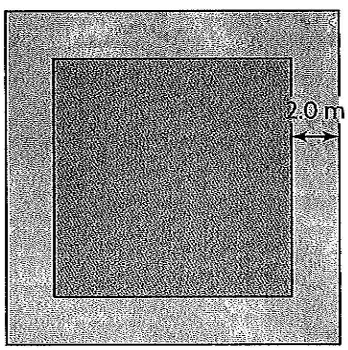
12. Algebraically determine the points of intersection of the parabolas
T $y = 2x^2 + 5x - 8$ and $y = -3x^2 + 8x - 1$.

13. Calculate the value of x.



14. A trained stunt diver is diving off a platform that is 15 m high into
A a pool of water that is 45 cm deep. The height, h, in metres, of the stunt diver above the water is modelled by $h = -4.9t^2 + 1.2t + 15$, where t is the time in seconds after starting the dive.
   a) How long is the stunt diver above 15 m?
   b) How long is the stunt diver in the air?

15. A rectangle is 5 cm longer than it is wide. The diagonal of the rectangle is 18 cm. Determine the dimensions of the rectangle.



16. A square lawn is surrounded by a concrete walkway that is 2.0 m wide, as shown at the left. If the area of the walkway equals the area of the lawn, what are the dimensions of the lawn? Express the dimensions to the nearest tenth of a metre.

17. Use a chart like the one below to compare the advantages of solving a quadratic equation by factoring and by using the quadratic formula. Provide an example of an equation that you would solve using each strategy.

Strategy	Advantages	Example
Factoring		
Quadratic Formula		

Extending

18. Determine the points of intersection of the line $y = 2x + 5$ and the circle $x^2 + y^2 = 36$.

19. Determine a quadratic equation, in standard form, that has each pair of roots.
   a) $x = -3$ and $x = 5$ b) $x = \frac{2 \pm \sqrt{5}}{3}$

20. Three sides of a right triangle are consecutive even numbers, when measured in centimetres. Calculate the length of each side.