5.5 Solving Problems Using Quadratic Relations

GOAL

Model and solve problems using the vertex form of a quadratic relation.

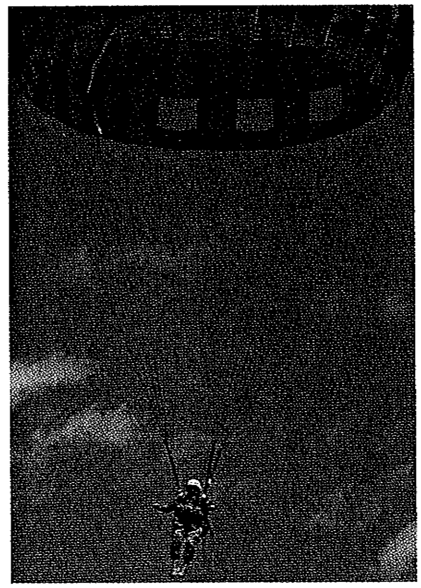
YOU WILL NEED

- grid paper
- ruler

LEARN ABOUT the Math

Smoke jumpers are firefighters who parachute into remote locations to suppress forest fires. They are often the first people to arrive at a fire. When smoke jumpers exit an airplane, they are in free fall until their parachutes open.

A quadratic relation can be used to determine the height, H, in metres, of a jumper t seconds after exiting an airplane. In this relation, $a = -0.5g$, where g is the acceleration due to gravity. On Earth, $g = 9.8\ \text{m/s}^2$.



Environment *Connection*

In a recent year, 3596 of the 7290 forest fires in Canada were caused by human activities such as careless smoking, campfires, use of welding equipment, or operation of a motor vehicle.

? If a jumper exits an airplane at a height of 554 m, how long will the jumper be in free fall before the parachute opens at 300 m?

EXAMPLE 1 Connecting information from a problem to a quadratic model

a) Determine the quadratic relation that will model the height, H, of the smoke jumper at time t.

b) Determine the length of time that the jumper is in free fall.

Conor's Solution

a) $H = a(t - h)^2 + k$ ← I decided to use the vertex form of the quadratic relation because the problem contains information about the vertex.

$H = a(t - 0)^2 + 554$

The vertex is the point at which the jumper exited the plane. So the vertex has coordinates (0, 554). I substituted these coordinates into the general equation.

$H = -0.5(9.8)(t - 0)^2 + 554$

$H = -4.9(t - 0)^2 + 554$ is an equation in vertex form for the quadratic relation that models this situation.

Since $a = -0.5g$ and $g = 9.8$ m/s^2, I substituted these values into the vertex form of the equation.

$H = -4.9t^2 + 554$ is an equation in standard form for the quadratic relation that models this situation.

I noticed that the value of a is the same in both vertex form and standard form. This makes sense because the parabolas would not be congruent if they were different.

b)

$$
\begin{aligned}
300 &= -4.9t^2 + 554 \\
-254 &= -4.9t^2 \\
\frac{-254}{-4.9} &= \frac{-4.9t^2}{-4.9} \\
51.84 &= t^2 \\
\sqrt{51.84} &= t
\end{aligned}
$$

Because the parachute opened at 300 m, I substituted 300 for H. Then I solved for t.

$7.2 \doteq t$, since $t > 0$

The jumper is in free fall for about 7.2 s.

In this situation, time can't be negative. So, I didn't use the negative square root of 51.84.

Reflecting

A. Why was zero used for the t-coordinate of the vertex?

B. How would the equation change if the jumper hesitated for 2 s before exiting the airplane, after being given the command to jump?

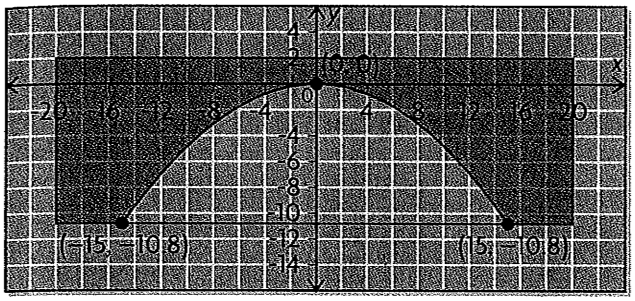
C. Why was the vertex form easier to use than either of the other two forms of a quadratic relation in this problem?

APPLY the Math

EXAMPLE 2 Solving a problem using a quadratic model

The underside of a concrete railway underpass forms a parabolic arch. The arch is 30.0 m wide at the base and 10.8 m high in the centre. The upper surface of the underpass is 40.0 m wide. The concrete is 2.0 m thick at the centre. Can a truck that is 5 m wide and 7.5 m tall get through this underpass?

Lisa's Solution



I started by drawing a diagram. I used a grid and marked the top of the arch as (0, 0). The upper surface of the underpass is 2 m above the top of the arch at the centre. The arch is 10.8 m high in the centre and 30 m wide at the base (or 15 m wide on each side). I marked the points (−15, −10.8) and (15, −10.8) and drew a parabola through these two points and the origin.

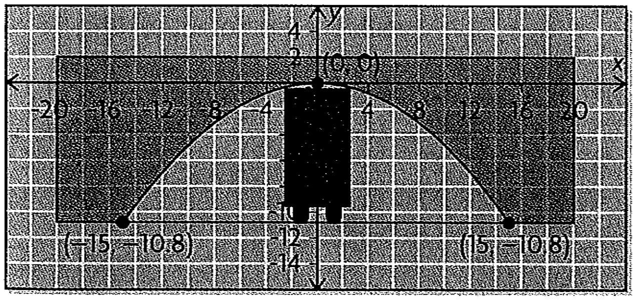
$$y = ax^2$$

The vertex of the parabola is at the origin, so I did not translate $y = ax^2$.

$$-10.8 = a(15)^2$$
$$-10.8 = 225a$$
$$-0.048 = a$$

I determined the value of a by substituting the coordinates of a point on the graph, (15, −10.8), into this equation. Then I solved for a.

$y = -0.048x^2$ is the quadratic relation that models the arch of the railway underpass.



The truck has the best chance of getting through the underpass if it passes through the centre. Since the truck is 5 m wide, this means that the position of the right corner of the truck has an x-coordinate of 2.5. I substituted $x = 2.5$ into the equation to check the height of the underpass at this point.

$$y = -0.048(2.5)^2$$
$$y = -0.048(6.25)$$
$$y = -0.3$$

$$\text{Height at } (2.5, -0.3) = 10.8 - 0.3 = 10.5$$

I determined the height from the ground at this point by subtracting 0.3 from 10.8.

The truck can get through. Since the truck is 7.5 m tall, there is 3 m of clearance.

The truck can get through the underpass, even if it is a little off the centre of the underpass.

EXAMPLE 3 Selecting a strategy to determine the vertex form

Write the quadratic relation $y = x^2 - 4x - 5$ in vertex form, and sketch the graph by hand.

Coral's Solution

$y = x^2 - 4x - 5$
$y = (x + 1)(x - 5)$

I rewrote the equation of the quadratic relation in factored form because I knew that I could determine the coordinates of the vertex from this form.

Zeros:
$0 = (x + 1)(x - 5)$
$x = -1$ and $x = 5$
The axis of symmetry is
$x = \dfrac{-1 + 5}{2}$ so $x = 2$.

I set $y = 0$ to determine the zeros. I used the zeros to determine the equation of the axis of symmetry.

$y = (2)^2 - 4(2) - 5$
$y = 4 - 8 - 5$
$y = -9$

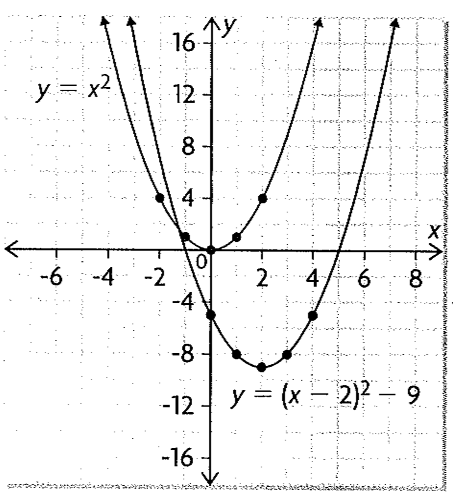
I substituted $x = 2$ into the standard form of the equation to solve for y.

The vertex is at $(h, k) = (2, -9)$.
The coefficient of x^2 is $a = 1$.

I knew that the value of a must be the same in the standard, factored, and vertex forms. If it were different, the parabola would have different widths.

The relation is $y = (x - 2)^2 - 9$ in vertex form.

I substituted what I knew into the vertex form, $y = a(x - h)^2 + k$.



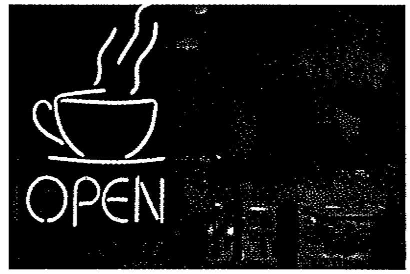
I sketched the graph of $y = x^2$ and translated each point 2 units right and 9 units down.

EXAMPLE 4 Representing a situation with a quadratic model

The Next Cup coffee shop sells a special blend of coffee for \$2.60 per mug. The shop sells about 200 mugs per day. Customer surveys show that for every \$0.05 decrease in the price, the shop will sell 10 more mugs per day.

a) Determine the maximum daily revenue from coffee sales and the price per mug for this revenue.

b) Write an equation in both standard form and vertex form to model this problem. Then sketch the graph.



Dave's Solution: Connecting the zeros of a parabola to the vertex form of the equation

a) Let x represent the number of \$0.05 decreases in price, where Revenue = (price)(mugs sold).

I defined a variable that connects the price per mug to the number of mugs sold.

$r = (2.60 - 0.05x)(200 + 10x)$

I used the information in the problem to write expressions for the price per mug and the number of mugs sold in terms of x. If I drop the price by \$0.05, x times, then the price per mug is $2.60 - 0.05x$ and the number of mugs sold is $200 + 10x$.

I used my expressions to write a relationship for daily revenue, r.

$0 = (2.60 - 0.05x)(200 + 10x)$
$2.60 - 0.05x = 0$, so $x = 52$
or
$200 + 10x = 0$, so $x = -20$

Since the equation is in factored form, the zeros of the equation can be calculated by letting $r = 0$ and solving for x.

$x = \dfrac{52 + (-20)}{2}$
$x = 16$

I used the zeros to determine the equation of the axis of symmetry.

$r = [2.60 - 0.05(16)][200 + 10(16)]$
$r = (1.80)(360)$
$r = 648$
The maximum daily revenue is \$648.

The maximum value occurs at the vertex. To calculate it, I substituted the x-value for the axis of symmetry into the revenue equation.

Price per mug for maximum revenue

$= 2.60 - 0.05(16)$

$= 1.80$

I substituted $x = 16$ into the expression for the price per mug.

The coffee shop should sell each mug of coffee for \$1.80 to achieve a maximum daily revenue of \$648.

b) In this relation, the maximum value is $r = 648$. It occurs when $x = 16$.

The vertex is $(16, 648) = (h, k)$.

$r = a(x - 16)^2 + 648$

The maximum value occurs at the vertex of a quadratic relation. To write the equation in vertex form, substitute the values of h and k into the vertex form of the general equation.

When $x = 0$,

$$r = (2.60)(200) = 520$$
$$520 = a(0 - 16)^2 + 648$$
$$520 = a(-16)^2 + 648$$
$$-128 = 256a$$
$$-0.5 = a$$

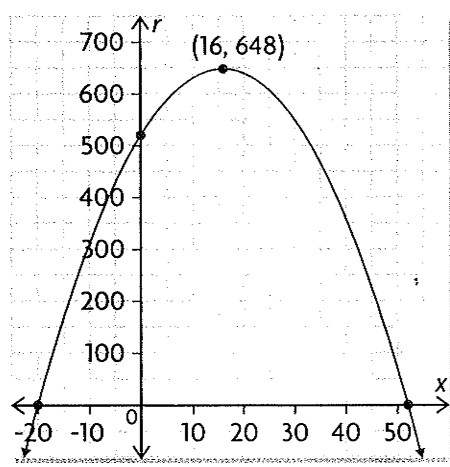
The equation in vertex form is $r = -0.5(x - 16)^2 + 648$.

Since the coffee shop sells 200 mugs of coffee when the price is \$2.60 per mug, the point (0, 520) is on the graph. I substituted these coordinates into the equation and solved for a.

$$r = -0.5(x - 16)^2 + 648$$
$$r = -0.5(x^2 - 32x + 256) + 648$$
$$r = -0.5x^2 + 16x - 128 + 648$$
$$r = -0.5x^2 + 16x + 520$$

I expanded to get the equation in standard form.

The equation in standard form is $r = -0.5x^2 + 16x + 520$.

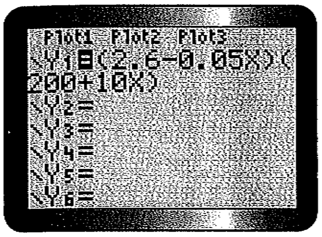


The vertex is at (16, 648). The zeros are at (52, 0) and (−20, 0).
The y-intercept is at (0, 520).

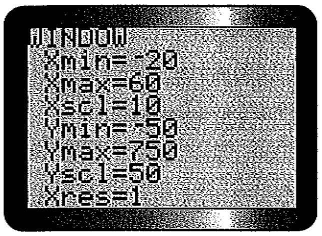
I used these points to sketch the graph of the relation.

Toni's Solution: Selecting a graphing calculator to determine the quadratic model

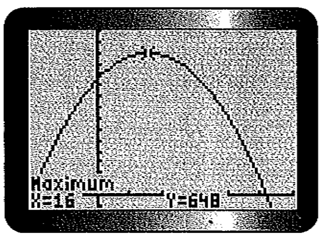
a)



Once I created the revenue equation, I entered it into the equation editor as Y1.



I graphed the revenue equation. I had to adjust the window settings until I could see the zeros and the vertex.



Since the vertex in this model represents the maximum value, I determined it using the maximum operation.

The vertex is at (16, 648).
The maximum daily revenue is \$648.
Selling price $= 2.60 - 0.05(16)$
$= 1.80$
Each mug of coffee should be sold for \$1.80 to maximize the daily revenue.

The maximum value is $y = 648$. This means that the maximum daily revenue is \$648. It occurs when $x = 16$.

b) The equation in standard form is
$y = (2.60 - 0.05x)(200 + 10x)$
$y = -0.5x^2 + 16x + 520$

Since the calculator has already produced the graph of the model, I only needed to determine the vertex form. I took the revenue equation and expanded it to get the equation in standard form.

$a = -0.5$
The vertex is at $(16, 648) = (h, k)$.
$y = a(x - h)^2 + k$
$y = -0.5(x - 16)^2 + 648$
This is the equation in vertex form.

Since the value of a is the same in all forms of a quadratic relation, I used it along with the coordinates of the vertex, and substituted to obtain the equation in vertex form.

Tech | *Support*
For help determining the maximum value of a relation using a TI-83/84 graphing calculator, see Appendix B-9. If you are using a TI-*n*spire, see Appendix B-45.

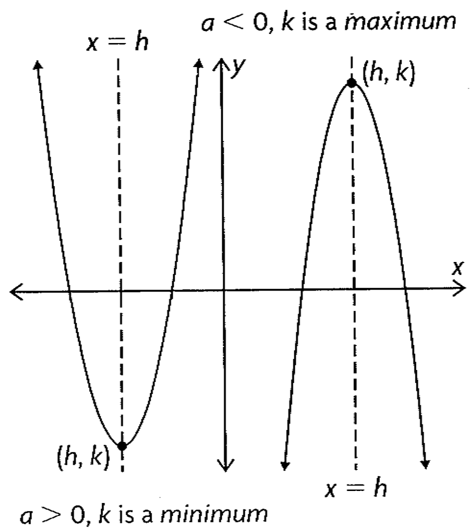
In Summary

Key Idea

- All quadratic relations can be expressed in vertex form and standard form. Quadratic relations that have zeros can also be expressed in factored form.
- For any parabola, the value of a is the same in all three forms of the equation of the quadratic relation.

Need to Know

- The y-coordinate of the vertex of a parabola represents the maximum or minimum value of the quadratic relation. The coordinates of the vertex are easily determined from the vertex form of the equation.
- If a situation can be modelled by a quadratic relation of the form $y = a(x - h)^2 + k$, the maximum or minimum value of y is k and it occurs when $x = h$.



- If $y = ax^2 + bx + c$ can be factored as a product of first-degree binomials and a constant, $y = a(x - r)(x - s)$, then this equation can be used to determine the vertex form of the quadratic relation as follows:
  - Use $x = \dfrac{r + s}{2}$ to determine the equation of the axis of symmetry. This gives you the value of h.
  - Substitute $x = \dfrac{r + s}{2}$ into $y = ax^2 + bx + c$ to determine the y-coordinate of the vertex. This gives you the value of k.
  - Substitute the values of a, h, and k into $y = a(x - h)^2 + k$.

CHECK Your Understanding

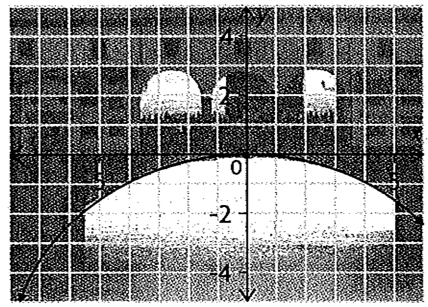
1. Use the given information to determine the equation of each quadratic relation in vertex form, $y = a(x - h)^2 + k$.

a) $a = 2$, vertex at $(0, 3)$ **c)** $a = -1$, vertex at $(3, -2)$

b) $a = -3$, vertex at $(2, 0)$ **d)** $a = 0.5$, vertex at $(-3.5, 18.3)$

2. Determine each maximum or minimum value in question 1.

3. The arch of the bridge in this photograph can be modelled by a parabola.



a) Determine an equation of the parabola.

b) On the upper part of the bridge, three congruent arches are visible in the first and second quadrants. What can you conclude about the value of a in the equations of the parabolas that model these arches? Explain.

PRACTISING

4. Determine the equation of a quadratic relation in vertex form, given the following information.

a) vertex at $(0, 3)$, passes through $(2, -5)$

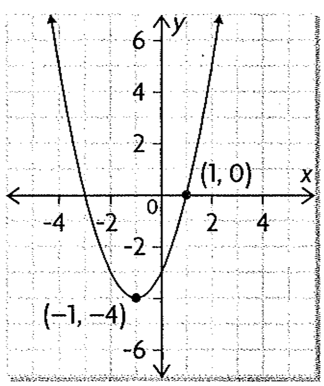
b) vertex at $(2, 0)$, passes through $(5, 9)$

c) vertex at $(-3, 2)$, passes through $(-1, 14)$

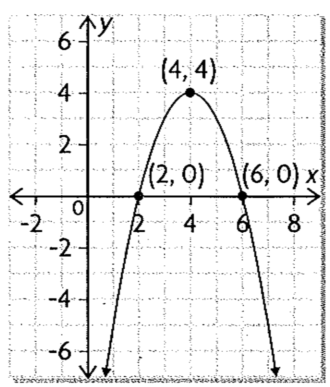
d) vertex at $(5, -3)$, passes through $(1, -8)$

5. Determine the equation of each parabola in vertex form.

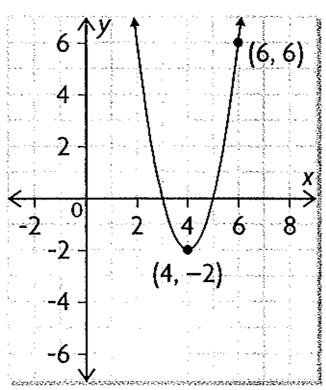
a)



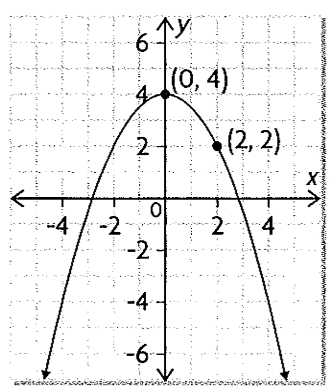
c)



b)



d)



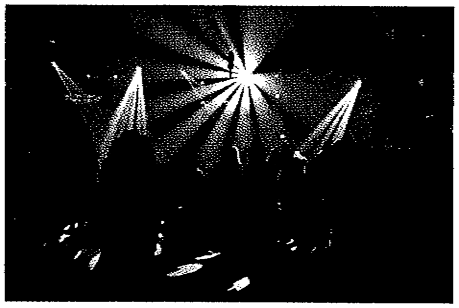
6. Write each equation in question 5 in standard form and factored form.

7. A quadratic relation has zeros at −2 and 8, and a y-intercept of 8.
K Determine the equation of the relation in vertex form.

8. The quadratic relation $y = 2(x + 4)^2 - 7$ is translated 5 units right and 3 units down. What is the minimum value of the new relation? Write the equation of this relation in vertex form.

9. Express each equation in standard form and factored form.
   a) $y = (x - 4)^2 - 1$
   b) $y = 2(x + 1)^2 - 18$
   c) $y = -(x + 5)^2 + 1$
   d) $y = -3(x + 3)^2 + 75$

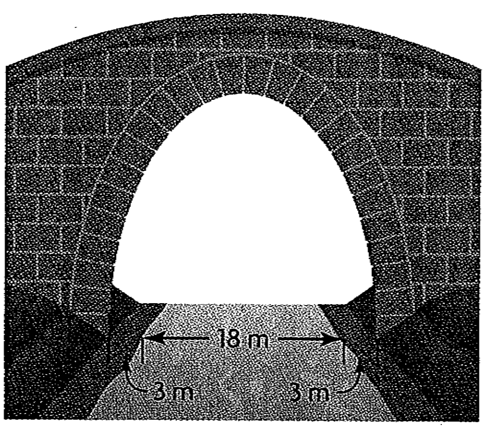
10. Express each equation in factored form and vertex form.
   a) $y = 2x^2 - 12x$
   b) $y = -2x^2 + 24x - 64$
   c) $y = 2x^2 - x - 6$
   d) $y = 4x^2 + 20x + 25$



11. A dance club has a \$5 cover charge and averages 300 customers on Friday nights. Over the past several months, the club has changed the cover price several times to see how this affects the number of customers. For every increase of \$0.50 in the cover charge, the number of customers decreases by 30. Use an algebraic model to determine the cover charge that maximizes revenue.

12. The graph of $y = -2(x + 5)^2 + 8$ is translated so that its new zeros are −4 and 2. Determine the translation that was applied to the original graph.

13. The average ticket price at a regular movie theatre (all ages) from 1995 to 1999 can be modelled by $C = 0.06t^2 - 0.27t + 5.36$, where C is the price in dollars and t is the number of years since 1995 ($t = 0$ for 1995, $t = 1$ for 1996, and so on).
   a) When were ticket prices the lowest during this period?
   b) What was the average ticket price in 1998?
   c) What does the model predict the average ticket price will be in 2010?
   d) Write the equation for the model in vertex form.



14. A bridge is going to be constructed over a river. The underside of the
A bridge will form a parabolic arch, as shown in the picture. The river is 18 m wide and the arch will be anchored on the ground, 3 m back from the riverbank on both sides. The maximum height of the arch must be between 22 m and 26 m above the surface of the river. Create two different equations to represent arches that satisfy these conditions. Then use graphing technology to graph your equations on the same grid.

15. A movie theatre can accommodate a maximum of 350 moviegoers per day. The theatre operators have been changing the admission price to find out how price affects ticket sales and profit. Currently, they charge \$11 a person and sell about 300 tickets per day. After reviewing their data, the theatre operators discovered that they could express the relation between profit, P, and the number of \$1 price increases, x, as $P = 20(15 - x)(11 + x)$.

a) Determine the vertex form of the profit equation.

b) What ticket price results in the maximum profit? What is the maximum profit? About how many tickets will be sold at this price?

16. The underside of a bridge forms a parabolic arch. The arch has a maximum height of 30 m and a width of 50 m. Can a sailboat pass under the bridge, 8 m from the axis of symmetry, if the top of its mast is 27 m above the water? Justify your solution.

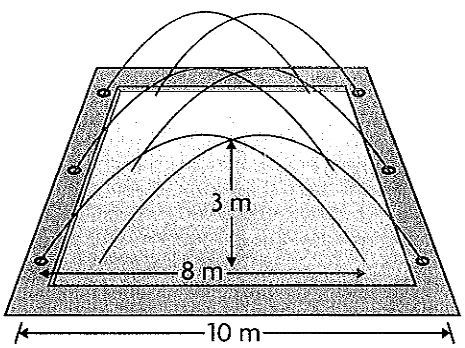
17. **T** A parabola has a y-intercept of -4 and passes through points $(-2, 8)$ and $(1, -1)$. Determine the vertex of the parabola.

18. **C** Serena claims that the standard form of a quadratic relation is best for solving problems where you need to determine the maximum or minimum value, and that the vertex form is best to use to determine a parabola's zeros. Do you agree or disagree? Explain.

Extending

19. The equation of a parabola is $y = a(x - 1)^2 + q$, and the points $(-1, -9)$ and $(1, 1)$ lie on the parabola. Determine the maximum value of y.

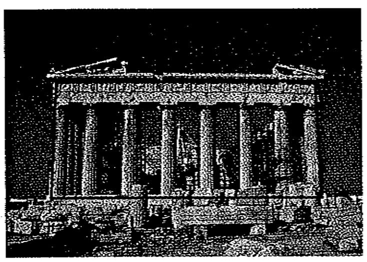
20. A rectangular swimming pool has a row of water fountains along each of its two longer sides. The two rows of fountains are 10 m apart. Each fountain sprays an identical parabolic-shaped stream of water a total horizontal distance of 8 m toward the opposite side. Looking from one end of the pool, the streams of water from the two sides cross each other in the middle of the pool at a height of 3 m.



a) Determine an equation that represents a stream of water from the left side and another equation that represents a stream of water from the right side. Graph both equations on the same set of axes.

b) Determine the maximum height of the water.

YOU WILL NEED
- graphing calculator

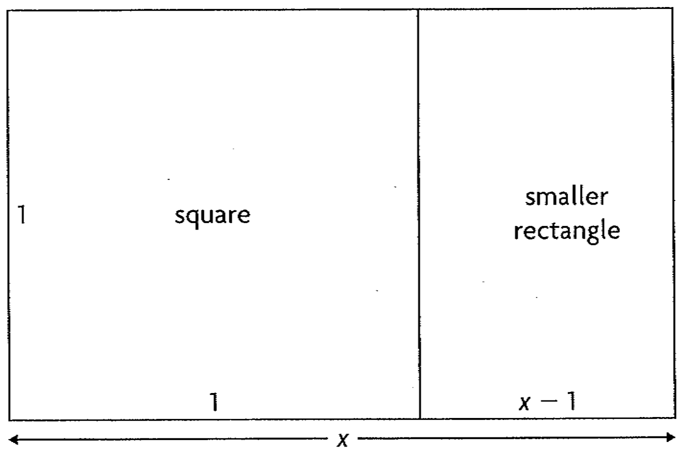


Curious | Math

The Golden Rectangle

The golden rectangle is considered one of the most pleasing shapes to the human eye. It is often used in architectural design, and it can be seen in many famous works of art. For example, the golden rectangle can be seen in Leonardo Da Vinci's *Mona Lisa* and in the *Parthenon* in Athens, Greece.

One of the properties of the golden rectangle is its dimensions. When it is divided into a square and a smaller rectangle, the smaller rectangle is similar to the original rectangle.



The ratio of the longer side to the shorter side in a golden rectangle is called the golden ratio.

If the length of the shorter side is 1 unit, and if x represents the length of the longer side, then $\frac{x}{1} = x$ is also the value of the golden ratio. A quadratic relation can be used to determine the value of the golden ratio.

1. Create a proportion statement to compare the golden ratio with the ratio of the lengths of the corresponding sides in the smaller rectangle.
2. Substitute the values in the diagram into your proportion statement. Then rearrange your proportion statement to obtain a quadratic relation.
3. Using graphing technology, graph the quadratic relation that corresponds to this equation.
4. What feature of the graph represents the value of the golden ratio?
5. Use graphing technology to determine the value of the golden ratio, correct to three decimal places.